Preprint of publication in J. Symbolic Computation (1997) **24**, 563–573

Rectifying Transformations for the Integration of Rational Trigonometric Functions

D.J.Jeffrey

Dept of Applied Mathematics, The University of Western Ontario, London, Canada

(Received 16 May 1996, Accepted 28 January 1997)

The paper presents four rectifying transformations that can be applied to the integration of a real rational expression of trigonometric functions. Integration is with respect to a real variable. The transformations remove, from the real line, discontinuities and singularities that would otherwise appear. If the integration is with respect to a complex variable, the transformations remain valid. In that case, they move singularities from the real line to elsewhere in the complex plane.

1. Introduction

Let $\psi, \phi \in \mathbb{R}[x, y]$ be polynomials over $\mathbb{R}$, the field of real numbers. A rational trigonometric function over $\mathbb{R}$ is a function of the form

$$T(\sin z, \cos z) = \frac{\psi(\sin z, \cos z)}{\phi(\sin z, \cos z)} \ . \tag{1.1}$$

The problem considered here is the integration of such a function with respect to a real

$\Re(z)$ is the real part of z. The integrand has simple poles when $\cos z = \frac{5}{4}$, at the points $z = \pm i \ln 2$. Corresponding logarithmic singularities must therefore be present in the expression on the right-hand side of (1.2). This is verified by observing that arctangent has logarithmic singularities at $\pm i$ and that $3 \tan \frac{1}{2}(\pm i \ln 2) = \pm i$. Also, since it is standard for arctangent to have branch cuts from $\pm i$ along the lines $\{\pm iy, y \in [1, \infty)\}$, and since $3 \tan \frac{1}{2}(\pm iy) = \pm 3i \tanh \frac{1}{2} y$, the right side of (1.2) has branch cuts along the set of lines $L_1 = \{\pm iy, y \geq \ln 2\}$. These branch cuts, being associated with the logarithmic singularities, are unavoidable. However, there are other singularities and branch cuts present in the same expression. Since $3 \tan \frac{1}{2}(\pi + iy) = 3i \coth \frac{1}{2} y$, the right-hand side of (1.2) has singularities at $\pm\pi$ and branch cuts extending from these along the set of lines $L_2 = \{\pm\pi + iy, y \in R\}$. Figure 1 shows this information graphically; the singularities are indicated by triangles and diamonds, the branch cuts L_1 are shown using heavy wavy lines, and the branch cuts L_2 are shown using hatched lines. The figure uses periodicity to extend beyond the interval discussed above.

Now consider the equation

$$\hat{I}_1(z) = \int \frac{3 \,\mathrm{d}z}{5 - 4\cos z} = z + 2 \arctan \frac{\sin z}{2 - \cos z} \ . \tag{1.3}$$

The right side of this equation has singularities at the points $z = \pm i \ln 2$ and branch cuts along the lines L_1, but it does not have singularities at $\pm\pi$, nor branch cuts along L_2. Therefore, if the integral were to be evaluated along a contour that cut one or more lines in L_2, the evaluation would be more efficient and reliable using (1.3). For contours along the real axis, (1.3) satisfies the definition of an integral on the real domain of maximum extent (Jeffrey 1993). The present paper gives an algorithm for obtaining (1.3) in place of (1.2).

Other examples show different forms of the problem. Consider

$$I_2(z) = \int \frac{(\cos z + 2\sin z + 1) \,\mathrm{d}z}{\cos^2 z - 2\sin z \cos z + 2\sin z + 3} = \arctan(\tan^2 \tfrac{1}{2} z + \tan \tfrac{1}{2} z) \ . \tag{1.4}$$

In the complex plane (again using periodicity to simplify the discussion), the integrand has poles at $z = 2\arctan\left(-\frac{1}{2} \pm \frac{1}{2}\sqrt{1 \pm 4i}\right)$. The right-hand side of (1.4) has d glarih- 3 5 2 mi sing

Recti

Definition: Let $\mathbb{D} \subset \mathbb{C}$ be a domain in the complex plane. Let a function $f : \mathbb{D} \to \mathbb{C}$ be defined and integrable everywhere on $\mathbb{D}$ except on a set of isolated singular points $\mathbb{S}_f$. Let $F, G : \mathbb{D} \to \mathbb{C}$ be primitives of f, which is to say, their derivatives satisfy $F' = G' = f$, and further let them have sets of singular points $\mathbb{S}_F$ and $\mathbb{S}_G$ respectively. Necessarily, we have $\mathbb{S}_F, \mathbb{S}_G \supseteq \mathbb{S}_f$. A transformation $\mathcal{T}$ is a rectifying transformation on $\mathbb{D}$ if it has the properties that $\mathcal{T}(F) = G$ and $\mathbb{S}_G \subset \mathbb{S}_F$; if $\mathbb{S}_G = \mathbb{S}_F$ then the transformation is neutral, and if $\mathbb{S}_G \supset \mathbb{S}_F$ then the transformation is exacerbating.
Example: It was shown by Jeffrey (1993) that given functions $A, B : \mathbb{R} \to \mathbb{R}$, and the integrand $f = A'/A + B'/B$, the transformation $\mathcal{T}(\ln A + \ln B) = \ln(AB)$ can be rectifying or neutral on $\mathbb{R}$. In the particular case in which $A = \sin x$ and $B = \csc x - \cot x$, and thus $f = \cot \frac{1}{2}x$, it is rectifying. In the case $A = 1/x$ and $B = 1/(1 + x)$, it is neutral. By reversing the roles of F and G, a transformation that is neutral or exacerbating is obtained.
Remark: Typically, the transformation $\mathcal{T}$ will be defined only on a specific and narrow class of functions.

The intention is that a rectifying transformation is used within an integration procedure after a primitive, or antiderivative, has been computed for a given integrand. The transformation then finds an equivalent antiderivative with better global properties. Therefore, before a description is given of the new transformations, the algorithm begins by reviewing an existing method for obtaining antiderivatives of trigonometric functions.

THEOREM 2.1. *Let $\phi, \psi \in \mathbb{Q}[x, y]$ be polynomials, where $\mathbb{Q}$ denotes the field of rational numbers, and let $T(\sin z, \cos z) = \phi(\sin z, \cos z)/\psi(\sin z, \cos z)$ be a e* where the ν_i are polynomials in u, possibly with complex coefficients. Let the functions ν_i be ordered so that those with purely real coefficients are numbered from 1 to m, for some m; these logarithms are then left unchanged. For $i > m$, the logarithms containing coefficients with nonzero imaginary parts are converted to arctangents using the rectifying transformation due to Rioboo (1991) that is described by Bronstein (1996) and also outlined by Geddes *et al.* (1992) in their exercise 11.18. By using the identity $\arctan(-f) = -\arctan f$, one can ensure that the leading coefficient of each P_j is positive, albeit, determining the sign of the leading coefficient might require considerable effort. $\square$

Remark: The rectifying transformations that are now described are independent of theorem 2.1, but they are most usefully applied to expressions such as (2.1), whether obtained by the method given in the proof, or some other. In particular, some CAS may obtain (2.2) with a partial fraction computation. Also, the theorem is stated for functions over $\mathbb{Q}$, but in many cases, a CAS will be able to integrate functions defined over extensions of $\mathbb{Q}$.

LEMMA 2.1.1. *For a polynomial $P \in \mathbb{R}[u]$ and u as above, define the transformation $\mathcal{E}$ by*

$$\mathcal{E}P(z) = \mathcal{E}[P(u)] = (\cos \tfrac{1}{2}z)^{\deg P} P(\tan \tfrac{1}{2}z) \ . \tag{2.3}$$

Then $\mathcal{E}P(2n\pi + \pi)$ is finite and $\mathcal{E}P(z)$ has the same zeros on $\mathbb{C}$ as $P(\tan \frac{1}{2}z)$.

PROOF. By periodicity, only the domain $-\pi < \Re z \leq \pi$ need be considered. Let the polynomial $P(u) = \sum^m p_i u^i$, where $m = \deg P$. Then $\mathcal{E}P(\pi) = p_m \neq 0$. Further, $P(\tan \frac{1}{2}\pi) \neq 0$ because it is unbounded, and so neither P nor $\mathcal{E}P$ is zero at $z = \pi$. Since $z \neq \pi$ implies $\cos \frac{1}{2}z \neq 0$, then $P(\tan \frac{1}{2}z) = 0 \iff \mathcal{E}P(z) = 0$. $\square$

In terms of this transformation, we now give the rectifying transformations that are the main results of this paper. First the logarithmic terms in (2.1) are considered.

THEOREM 2.2. *For monic polynomials $\nu_i \in \mathbb{R}[u]$, let a primitive $F(z) = \sum \alpha_i \ln \nu_i(u)$ with $u = \tan \frac{1}{2}z$ as before. Then the transformation*

$$\mathcal{T}_1 F(z) = \mathcal{T}_1 \left[\sum \alpha_i \ln \nu_i(u)\right] = \sum \alpha_i \ln \mathcal{E}\nu_i(z) - \sum \alpha_i$$

from it.

$$\frac{z}{2} + \arctan\frac{(a-1)u+b}{1+au^2+bu} - \arctan K = \frac{z}{2} + \arctan\frac{b-K+(a-1-Kb)u-Kau^2}{1+Kb+(Ka-K+b)u+au^2} \ .$$

A value of K must now be found that ensures that the denominator is positive. The minimum value of the denominator is given by

$$\inf_{u\in\mathbb{R}} \left(1+Kb+(Ka-K+b)u+au^2\right) = 1+Kb-(Ka-K+b)^2/4a \ ,$$

and this must be greater than zero. The largest value of the infimum is obtained by setting $K = b(1+a)/(1-a)^2$, but using this value leads to a more complicated expression than the one in (2.6). The infimum is also positive for $K = b/(1+a)$ and using this leads to (2.6), after using $u = \sin z/(1+\cos z)$. □

We finally come to the general odd-degree case. The technique just used, of combining the arctangent with a constant, can be shown by counter example to fail. Thus the straightforward generalization would be to try to rectify an odd degree poynomial $P(u)$ by writing

$$\arctan P(u) - \arctan u - \arctan K = \arctan\frac{P-u-K(1+uP)}{1+uP+K(P-u)} \ .$$

Consider, however, the example $P = \frac{1}{10}u^3 - 4u$. The denominator then becomes equal to $\frac{1}{10}u^4 + \frac{1}{10}Ku^3 - 4u^2 - 5Ku + 1$. No value of K exists that makes this positive for all u. For example, the expression is always negative at $u = \frac{1}{10} - \frac{5}{8}K + \frac{1}{8}\sqrt{25K^2+16}$, if $K \geq 0$, and at a similar place if $K < 0$. Hence a more elaborate construction is needed.

3. Implementation and examples

The algorithm is now summarized.

1 The system identifies an integrand as rational trigonometric over $\mathbb{R}$ (in variable z let us say) and tries any preferred simpler evaluation strategies, for example, a sine or cosine substitution.
2 The system substitutes $u = \tan \frac{1}{2}z$ and passes the resulting rational function to its integrator. This step can be tried even if the coefficients are in $\mathbb{R}$ rather than $\mathbb{Q}$, because the integrator may succeed.
3 If an expression is returned in the form (2.1), then the algorithm can proceed. If it is not of this form (for example, it contains complex logarithms or arctangents of rational functions), then the algorithm fails.
4 For each term matching one of the four patterns, the rectifying transform is applied. Notice that the application of the transformation does not require any analysis or knowledge of the whole integral expression, but can be applied immediately upon recognising the pattern.
5 Any residual terms in u are returned to z.

This algorithm is now applied to the examples given in the introduction and to other examples. Starting with (1.2), the transformation $\mathcal{T}_3$ gives (1.3), by simple substitution in (2.6). To apply $\mathcal{T}_2$

present at $z = \pi$.

$$\int \frac{(7\cos z + 2\sin z + 3)\,\mathrm{d}z}{3\cos^2 z - \sin z\cos z + 4\cos z - 5\sin z + 1} = \ln\frac{3 + \sin z + \cos z}{\cos x - 2\sin x + 1}\ . \tag{3.3}$$

The singularity cannot be removed because it is induced by a pole in the integrand. Even so, the integral is in a form that is more convenient for further analysis than the original expression. Therefore, a CAS would do well to implement the transformation, independently of whether the transformation is rectifying. In this regard, it can be noticed that the transformation is linear in its argument, and even though theorem 2.2 was stated for a sum of logarithms, the transformation can be applied independently to each logarithm. Also, the polynomial arguments of the logarithms do not have to be monic, although again that is how the theorem was stated.

An example of the application of $\mathcal{T}_4$ is

$$\int \frac{(5\cos^2 z + 4\cos z - 1)\,\mathrm{d}z}{4\cos^3 z - 3\cos^2 z - 4\cos z - 1} = 2\arctan(\tan^3 \tfrac{1}{2}z - 2\tan\tfrac{1}{2}z)\ .$$

A value of $k > -\inf(u^4 - 2u^2) = 1$ must be selected. Since k cannot equal 1, the obvious choice is 2. This gives

$$\int \frac{(5\cos^2 z + 4\cos z - 1)\,\mathrm{d}z}{4\cos^3 z - 3\cos^2 z - 4\cos z - 1} = z - 2\arctan\frac{7\cos z\sin z + 3\sin z}{5\cos^2 z + 2\cos z + 1} - 2\arctan\frac{\sin z}{3 + \cos z}.$$

If it is possible to choose $k = 1$, this is clearly the best choice, since one of the arctangent terms is then removed, as the next example shows. We have6' e t) l 2iuh4 i) 2an

Here the $\mathcal{T}_3$ transformation has already been applied. If the substitution $u = \tan z$ is used instead of $u = \tan \frac{1}{2} z$, only a single arctangent is needed. Within the present scheme, the