This paper was published in ACM Transactions on Mathematical Software Vol. 20 (1994) 124 –135. The pagination of this version matches the original, but the line breaks do not.

The Evaluation of Trigonometric Integrals Avoiding Spurious Discontinuities

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be integrated using algorithms such as those by Hermite, Horowitz and Rothstein—Trager [Geddes *et al.* 1992]. Even for integrands that are not rational functions of $\sin x$ and $\cos x$, the transformed integral in u can sometimes be evaluated by the system. Although the substitution is a well-established topic in calculus textbooks, all published accounts of it — and hence almost all implementations in computer algebra systems — are essentially incomplete, because they fail to address discontinuity problems in the antiderivatives obtained.

The unsatisfactory aspect of the existing theory can be quickly established by an example. Consider the problem of integrating $3/(5 -$

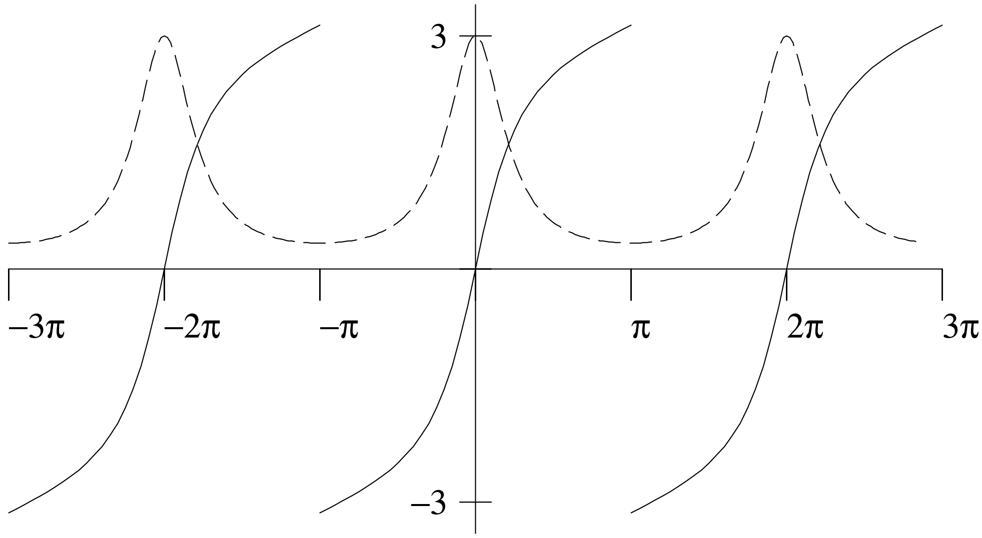

Fig 1. The integrand and integral given in eq. (1). — — —, integrand. ———, integral.

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taught in calculus books and commonly implemented by computer algebra systems.

Standard algorithm. Given an integrable function $f(\sin x, \cos x)$ whose indefinite integral is required, make the transformation

$$\int f(\sin x, \cos x)\,\mathrm{d}x = \int f\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{2}{1+u^2}\,\mathrm{d}u.$$

Evaluate the right-hand side using standard algorithms and substitute $u = \tan\frac{1}{2}x$ into the expression obtained to get the final result.

Here we modify this algorithm so that the result does not contain spurious discontinuities (spurious in the sense defined above).

2. AN EXAMPLE OF A CONTINUOUS ANTIDERIVATIVE.

In this section, we introduce the algorithm by showing that we can derive a replacement for (1) that is continuous everywhere on the real line. We first replace the indefinite integral in (1) with a definite integral, because indefinite integrals do not show the domain of the integration variables. Thus we consider

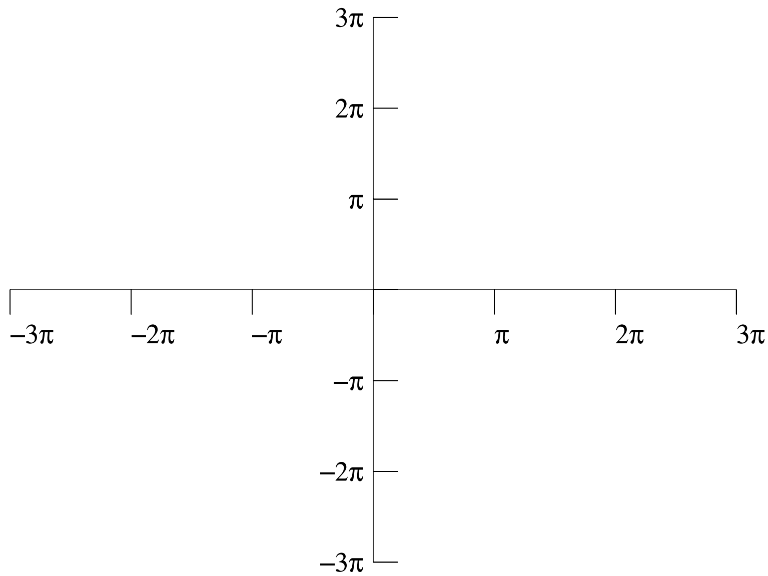
$$g(x) = \int_{-\pi}^{x} \frac{3\,\mathrm{d}\theta}{5 - 4\cos\theta}\ .$$

By rewriting Weierstrass's substitution for this integral in the form

$$\theta = 2\arctan u\ ,$$

and recalling that the range of arctan is the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we see that we must restrict θ to lie in the interval $(-\pi, \pi)$, and this requires that the integral be restricted to $|x| < \pi$. To evaluate the integral for $|x| > \pi$, we find an integer n such that $(2n-1)\pi < x < (2n+1)\pi$. Then

$$\int_{-\pi}^{x} \frac{3\,\mathrm{d}\theta}{5-4\cos\theta} = \int_{-\pi}^{(2n-1)\pi} \frac{3\,\mathrm{d}\theta}{5-4\cos\theta} + \int_{(2n-1)\pi}^{x} \frac{3\,\mathrm{d}\theta}{5-4\cos\theta}\ .$$


3π
2π
π
−3π
−2π
−π
π
2π
3π
−π
−2π
−3π

substitution is better considered as a family of substitutions; secondly, the discontinuities that we need to remove are introduced by the *method* of integration, and are not a property of the integrand; thirdly, not all discontinuities are spurious; and lastly, the discontinuities can be present even in expressions that do not contain arctangents.

So that there is no possibility of the equations that follow being misunderstood, we shall adopt the following notation. When an integral is simplified to a form containing spurious discontinuities, we shall use the symbol X| R

The discontinuities at $x = (n + \frac{1}{}$

and it will be continuous by the fundamental theorem; the function we actually find, however, is $\hat{g}$.

$$\hat{g}(x) = \int f(\phi(x))\phi'(x)\,\mathrm{d}x \ .$$

So long as $x < b$, the function g can be expressed, by the above theorem, in terms of $\hat{g}$.

$$g(x) = \hat{g}(x) - \hat{g}(a) \ .$$

For $x > b$, the connection between g and $\hat{g}$ is obtained as follows.

$$\begin{aligned} g(x) = \int_a^x f(s)\,\mathrm{d}s &= \int_a^c f(s)\,\mathrm{d}s - \int_x^c f(s)\,\mathrm{d}s \\ &= \int_a^c f(s)\,\mathrm{d}s - \int_{\phi^{-1}(x)}^{\phi^{-1}(c)} f(\phi(t))\phi'(t)\,\mathrm{d}t \\ &= g(c) - \hat{g}(c) + \hat{g}(x) \ . \end{aligned}$$

To eliminate $g(c)$ from this equation, we calculate

$$\begin{aligned} \lim_{x\to b-} \hat{g}(x) - \lim_{x\to b+} \hat{g}(x) &= \lim_{x\to b-} (g(x) + \hat{g}(a)) - \lim_{x\to b+} (g(x) - g(c) + \hat{g}(c)) \\ &= g(c) - \hat{g}(c) + \hat{g}(a) \ , \end{aligned}$$

where the limits have been evaluated using the continuity of g at b. We can combine the expressions for g in the intervals $[a, b]$ and $[b, c]$ into a single equation using the Heaviside, or step, function H.

$$g(x) = \hat{g}(x) - \hat{g}(a) + H(x-b)\left[\lim_{x\to b-} \hat{g}(x) - \lim_{x\to b+} \hat{g}(x)\right] \ .$$

This gives a continuous expression for the desired function g in terms of the computable function $\hat{g}$. In graphical terms, the functions g and $\hat{g}$ will separate from each other by a series of jumps at isolated points. The continuous function g can be built from $\hat{g}$ by cancelling the jumps.

For the case of the substitution $\phi(x) = \tan \frac{1}{2}x$, the isolated points at which the substitution might introduce discontinuities are those where the tangent becomes unbounded, namely $x = (2n+1)\pi$, n being an integer. Thus integrals obtained using this substitution need to check the continuity only at $x = (2n+1)\pi$. In addition, since $\tan \frac{1}{2}x$ has period 2π, the substitution can only be applied to integrands having the same period, meaning that all jumps will be equal. Thus we need to examine only one point, say $x = \pi$, to obtain the size of the jump. Because of the periodicity, the jumps can be cancelled using the floor function $\lfloor x \rfloor$. If $\hat{g}(x)$ is a provisional antiderivative that has been obtained using the $\tan \frac{1}{2}x$ substitution, we calculate

$$K = \lim_{x\to \pi-} \hat{g}(x) - \lim_{x\to \pi+} \hat{g}(x) \ . \tag{9}$$

The function $\hat{g}(x) + K\lfloor (x+\pi)/2\pi \rfloor$ is then a corrected antiderivative of the given function that contains no spurious discontinuities and is discontinuous

Table 1. Functions $u = \phi(x)$ used in the Weierstrass algorithm and their corresponding substitutions.

Choice	$\phi(x)$	$\sin x$	$\cos x$	$\mathrm{d}x$	b	p
(a)	$\tan \frac{1}{2}x$	$\frac{2u}{1+u^2}$	$\frac{1-u^2}{1+u^2}$	$\frac{2\,\mathrm{d}u}{1+u^2}$	π	2π
(b)	$\tan(\frac{1}{2}x + \frac{1}{4}\pi)$	$\frac{u^2-1}{u^2+1}$	$\frac{2u}{u^2+1}$	$\frac{2\,\mathrm{d}u}{1+u^2}$	$\frac{1}{2}\pi$	2π
(c)	$\cot \frac{1}{2}x$	$\frac{2u}{1+u^2}$	$\frac{u^2-1}{1+u^2}$	$\frac{-2\,\mathrm{d}u}{1+u^2}$	0	2π
(d)	$\tan x$	$\frac{u}{\sqrt{1+u^2}}$	$\frac{1}{\sqrt{1+u^2}}$	$\frac{\mathrm{d}u}{1+u^2}$	$\frac{1}{2}\pi$	π

only when there is a singularity in the integrand. If either of the limits fail to exist, then a singularity of the integral coincides with the point, and no correction is needed.

5. COMPLETE STATEMENT OF THE ALGORITHM.

A complete statement of the modified algorithm, taking into account the points established in section 3 now follows. We start by defining a Weierstrass substitution to be one that uses a function appearing in table 1. There are other trigonometric substitutions used by computer algebra systems, sine and cosine being the obvious examples, but since these are never singular, they cannot lead to the problems addressed by this paper, and hence we have not included them in our definition.

Given an integrable function $f(\sin x, \cos x)$ whose indefinite integral is required, select one of the substitutions listed in table 1. The choice will be based on heuristics and since it affects only the form of the final result, it is not important to develop an extensive set of heuristics. However, we can note that choice (a) is good for integrands not containing $\sin x$, choice (b) is good for integrands not containing $\cos x$, (c) can be useful in cases in which (a) gives an integral that cannot be evaluated by the system, and (d) is good under conditions described in Gradshteyn and Ryzhik (1979, section 2.50). The integral is now transformed using the entries in the table. For example, for choice (c) we have

$$\int f(\sin x, \cos x)\,\mathrm{d}x = \int f\left(\frac{2u}{1+u^2}, \frac{u^2-1}{1+u^2}\right)\frac{-2\,\mathrm{d}u}{1+u^2} .$$

The integral in u is now evaluated using the standard routines of the system, and then u substituted for. Call the result $\hat{g}(x)$. Next we calculate

$$K = \lim_{x\to b-} \hat{g}(x) - \text{ laluated using the standard}$$

where fl CCSpoint

Finally, we do not need to correct (6), but if choice (d) were used, however unwisely, we continuous result for the integral (using Rioboo's algorithm or another method), we obtain

$$2\sqrt{2}\int\frac{1+\cos x}{1+\cos^2 x}dx = \ln\frac{\tan^2\frac{1}{2}x+\sqrt{2}\tan\frac{1}{2}x+1}{\tan^2\frac{1}{2}x-\sqrt{2}\tan\frac{1}{2}x+1}$$
$$+2\arctan(\sqrt{2}\tan\tfrac{1}{2}x+1)+2\arctan(\sqrt{2}\tan\tfrac{1}{2}x-1)+4\pi\lfloor(x-\pi)/2\pi\rfloor\ .$$

This reminds us that the Weierstrass algorithm is not the only one that can introduce spurious discontinuities into expressions for indefinite integrals. All such algorithms should be provided the arctangent is defined according to Kahan, and

$$\operatorname{csgn}(z) = \begin{cases} 1\,, & \text{for } \Re(z) > 0 \text{ or } (\Re(z) = 0 \text{ and } \Im(z) > 0) \\ -1\,, & \text{for } \Re(z) < 0 \text{ or } (\Re(z) = 0 \text{ and } \Im(z) < 0) \\ 0\,, & \text{for } z = 0. \end{cases}$$

As a result of recent changes in textbooks on introductory calculus, which now omit the Weierstrass substitution, users will be less likely to know the substitution. This may not be such a bad thing, since the treatment in the books was always misleading. Not only did the question of the continuity of the integral pass unmentioned, the exercises always avoided situations which might alert the reader to the flaw in the treatment. Users of computer systems who check results obtained from their systems against the standard published tables of integrals should be aware that the tables continue to contain incorrect entries.

Acknowledgements.

We gratefully thank Larry Gilligan, Jerry Glynn, W. Kahan, Robert S. Smith, Ralph Freese, M.A.H. Nerenberg and R.M. Corless for useful discussions on the subject matter of this paper.

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