Function evaluation on branch cuts

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1 Introduction

Once it is decided that a CAS will evaluate multivalued functions on their principal branches, questions arise concerning the branch definitions. The first questions concern the standardization of the positions of the branch cuts. These questions have largely been resolved between the various algebra systems and the numerical libraries, although not completely. In contrast to the computer systems, many mathematical textbooks are much further behind: for example, many popular textbooks still specify that the argument of a complex number lies between 0 and 2π. We do not intend to discuss these first questions here, however. Once the positions of the branch cuts have been fixed, a second set of questions arises concerning the evaluation of functions on their branch cuts.

In [2], Kahan considered the closure problem from several points of view and discussed different possible solutions. One of his proposals was a principle called counter clockwise continuity (CCC) for the determination of the closure of the elementary functions. To determine closure for any branch, one imagines circling the branch point counterclockwise (anticlockwise in the British hemisphere) and the closure is on the side one arrives at by this process. Thus, one decides $\arctan(5i/3) = \frac{1}{2}\pi + i\ln 2$ and $\arcsin(5/4) = \frac{1}{2}\pi - i\ln 2$. This convention has not been followed by all systems. In particular, Derive defines inverse tangent using clockwise continuity (CC), and therefore obtains $\arctan(5i/3) = -\frac{1}{2}\pi + i\ln 2$. There are many other cases within the current CAS of non-CCC closures.

Examining the reasoning behind the Derive selection of the closure of arctangent introduces the reasons for many of the departures from CCC in other systems. If both arcsine and arctangent are closed CCC, then the relation between them that is valid over the whole of $\mathbb{C}$ is

$$\begin{aligned}\arcsin z &= \arctan\frac{z}{\sqrt{1-z^2}} - \pi \quad (-\ln(1+z)) \\ &\quad +\pi \quad (-\ln(1-z)) \ , \end{aligned} \tag{1}$$

where   is the unwinding number [1]. Another way to express this uses complex conjugation:

$$\arcsin z = \overline{\arctan\left(\frac{\overline{z}}{\sqrt{1-\overline{z}^2}}\right)} \ , \tag{2}$$

where $\overline{z}$ is the complex conjugate of z. In contrast, if arctangent is defined as clockwise continuous (CC) the relation is

$$\arcsin z = \arctan\frac{z}{\sqrt{1-z^2}} \ . \tag{3}$$

This last relation is of prime importance within Derive, and overrides the importance of maintaining CCC.

Kahan recognized the importance of algebraic identities and their dependence upon closure definitions. For this reason, he also discussed the idea of a signed zero. We shall discuss both ideas here.

2 Closure by CCC

We start by considering whether CAS should follow Kahan in making all their functions CCC. The principles seem to be as follows.

1. Standards are good. We should like all our arctangents to simplify to the same thing on all systems.

2. CCC looked the best candidate for a standard. Before settling on CCC, Kahan had reviewed the implementations existing at the time of his article. As we have noted above, time has not established CCC as the standard, certainly within CAS, although it continues to exert a strong influence.

3. The standard allows the closure of new functions to be determined automatically. It should be noted that a finite branch cut cannot be CCC at both ends, unless it has a singularity in the cut. Therefore this principle will not always apply.

4. Any convention will allow some algebraic relations and invalidate others. This point has two consequences. First, one should not waste time searching

Thus we propose that $\ln(-1)$ simplify to $\pm\pi i$ instead of πi. Notice that we are not proposing to return a set of values, elementary functions do not return sets. Rather, the value returned can be operated on by mathematical operators the same way numbers can: for example, $(\pm i\pi)^2$ simplifies to $-\pi^2$.

The following table gives, for each of the multivalued elementary functions, the position of the branch cut, in DERIVE's implementation. Other CAS may place some of the branch cuts in different places. For example, the branch cuts of acoth were recently changed in Maple. The branch cuts are specified using $z = x + iy$. The table also contains for each function a relation that is valid on the whole complex plane except on the branch cuts, if a particular value is selected.

Function	Branch cut	Symmetry relation
ln :	$x < 0, y = 0$	$\ln \overline{z} = \overline{\ln z}$
asin :	$\lvert x\rvert > 1, y = 0$	$\text{asin}\, \overline{z} = \overline{\text{asin}\, z}$
acos :	$\lvert x\rvert > 1, y = 0$	$\text{acos}\, \overline{z} = \overline{\text{acos}\, z}$
atan :	$\lvert y\rvert > 1, x = 0$	$\text{atan}\, z^* = (\text{atan}\, z)^*$
acot :	$\lvert y\rvert > 1, x = 0$	$\text{acot}\, z^* = \pi + (\text{acot}\, z)^*$
asec :	$\lvert x\rvert < 1, y = 0$	$\text{asec}\, \overline{z} = \overline{\text{asec}\, z}$
acsc :	$\lvert x\rvert < 1, y = 0$	$\text{acsc}\, \overline{z} = \overline{\text{acsc}\, z}$
atanh :	$\lvert x\rvert > 1, y = 0$	$\text{atanh}\, \overline{z} = \overline{\text{atanh}\, z}$
acoth :	$\lvert x\rvert < 1, y = 0$	$\text{acoth}\, \overline{z} = \overline{\text{acoth}\, z}$
asinh :	$\lvert y\rvert > 1, x = 0$	$\text{asinh}\, z^* = (\text{asinh}\, z)^*$
acosh :	$x < 1, y = 0$	$\text{acosh}\, \overline{z} = \overline{\text{acosh}\, z}$
asech :	$x < 0, x > 1, y = 0$	$\text{asech}\, \overline{z} = \overline{\text{asech}\, z}$
acsch :	$\lvert y\rvert < 1, x = 0$	$\text{acsch}\, z^* = (\text{acsch}\, z)^*$
nth root :	$x < 0, y = 0$	$(\overline{z})^{1/n} = \overline{z^{1/n}}$

The ramifications on a CAS that systematically implemented this proposal are significant, but we think beneficial. For example, the last line of the above table means that $\sqrt{-1}$ should simplify to $\pm i$ instead of i. Otherwise, a CAS should not simplify $\sqrt{\overline{z}} - \overline{\sqrt{z}}$ to 0, even though z could be real and negative.

Although our reasoning may be valid, we realize that simplifying $\sqrt{-1}$ to $\pm i$ would have the mathematics community howling. After all, i is widely defined as being $\sqrt{-1}$. The problem comes from trying to define i in