The Complete Root Classification of a Parametric Polynomial on an Interval

ABSTRACT

Given a real parametric polynomial $p(x)$ and an interval $(a, b) \subset \mathbb{R}$, the Complete Root Classification (CRC) of $p(x)$ on (a, b) is a collection of all possible cases of its root classification on (a, b), together with the conditions its coefficients must satisfy for each case. In this paper, a new algorithm is proposed for the automatic computation of the complete root classification of a parametric polynomial on an interval. As a direct application, the new algorithm is applied to some real quantifier elimination problems.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms — *Algebraic algorithms*

General Terms

Algorithms

Keywords

Complete root classification, real root, parametric polynomial, interval, real quantifier elimination

1. INTRODUCTION

The counting and classifying of the roots of a polynomial have been the subject of many investigations. This paper concerns the complete root classification of a parametric polynomial on an interval.

RC and CRC. Let $p(x)$ be a real polynomial with constant coefficients. The *root classification (RC) of* $p(x)$ *on* $\mathbb{R}$ is denoted by

$$[L_1, L_2] = [[n_1, n_2, \ldots], [m_1, -m_1, m_2, -m_2, \ldots]],$$

where n_k are the multiplicities of the distinct real roots of $p(x)$ on $\mathbb{R}$, and m_k are the multiplicities of the distinct complex conjugate pairs of $p(x)$, and $L_1 = [n_1, n_2, \ldots]$ is called the *real RC of* $p(x)$ *on* $\mathbb{R}$. Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. The *RC of* $p(x)$ *on* (a, b) is denoted by a list $L = [n_1, n_2, \ldots]$, where $n_1, n_2, \ldots$ are the multiplicities of the distinct real roots of $p(x)$ on (a, b). For a real polynomial $p(x)$ with parametric coefficients, the *complete root classification (CRC) of* $p(x)$ *on* (a, b) is a collection of all possible cases of its RC on (a, b), together with the conditions its coefficients must satisfy for each case.

The history of CRC is short. The CRC of a real parametric quartic polynomial on R was found by Arnon in 1988 [2]; the first method for establishing the CRC of a real parametric polynomial of any degree on R was given by Yang, Hou and Zeng in 1996 [10]. They illustrated their method by computing the CRC of a reduced sextic polynomial. The first automatic generation of CRCs was described and implemented by Liang and Zhang [7], with some improvements added in [5]. Further improvements to the algorithm were made in [6] by replacing the 'revised sign lists' (Definition 4 below) with the direct use of 'sign lists'. As well as offering greater efficiency, the new algorithm offers a better filter for eliminating non-realizable conditions.

All works above are on R, and applications often need CRC on an interval. For example, in robust control [1] and problems concerning program termination [12], we have to determine the conditions on the parametric coefficients of $p(x)$ such that $\forall x > 0, p(x) > 0$, or the conditions such that $\forall x \in (a, b), p(x) \neq 0$. Therefore, it is meaningful to develop an algorithm for computing the CRC of a parametric polynomial on an interval.

However, in order to develop such an algorithm, we have to face two challenging problems. The first problem is the determination of the conditions for a parametric polynomial having a given number of real roots on an interval. One naturally thinks of the well-known Sturm sequence. The Sturm sequence of a polynomial with known, constant coefficients is a good tool for computing the number of real roots on an interval, but it is inconvenient and inefficient when the given polynomial has parametric coefficients. A better solution uses the fact that we know how to determine the conditions for a polynomial having a given number of real roots on R [6], and converts the problem of the determination of conditions on an interval into a problem on R. This is done in Section 3, where Theorem 4 is given. Let $p \in \mathbb{R}[x]$ with $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ and $a_n \neq 0$. Let $a, b \in \mathbb{R}$ such that $p(a) \neq 0$ and $p(b) \neq 0$. Let $\Psi_1(x) = (1 -$

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$(1+x^2)^n p\left(\frac{b+ax^2}{1+x^2}\right)$. Then $L = [r_1, r_2, \ldots, r_k]$ is the RC of $p(x)$ on (a, b), if and only if $L_2 = [r_1, r_1, r_2, r_2, \ldots, r_k, r_k]$ is the real RC of $\psi_2(x)$ on $\mathbb{R}$. Therefore, the conditions for $p(x)$ having L as its RC on an interval can be obtained by computing the conditions for $\psi_2(x)$ having L_2 as its real RC on $\mathbb{R}$.

The second problem is the computation of the Δ-sequence of ψ_2 (Definition 5). We try to determine the conditions for ψ_2 having L_2 as its real RC on $\mathbb{R}$. The set of all possible sign lists of ψ_2 can be determined by Theorem 2 and 3. Now, in order to make the multiplicities of the $2k$ distinct real roots of ψ_2 be $r_1, r_1, r_2, r_2, \ldots, r_k, r_k$ respectively, we also have to determine the possible sign lists of the polynomials in the Δ-sequence of ψ_2. According to Proposition 1, $\Delta^1(\psi_2)$ can be determined by the maximal index ℓ of non-vanishing members in the sign list of ψ_2 which actually is the total number of distinct (real and complex) roots of ψ_2. Since L_2 does not contain information about the number of distinct complex-conjugate roots of ψ_2, the maximal index ℓ is not uniquely determined. Therefore, unlike the case of RC on $\mathbb{R}$ [6], there may be more than one $\Delta^1(\psi_2)$ for the real RC L_2, and consequently the conditions for $\psi_2(x)$ having L_2 as its real RC on $\mathbb{R}$ would be more complicated. So the question is how to determine these $\Delta^1(\psi_2)$ and corresponding conditions.

In this paper, a new algorithm for the automatic computation of the CRC of a parametric polynomial on an interval is proposed. The new algorithm has been implemented in Maple. As an immediate application, the new algorithm has been applied to some real quantifier elimination problems. However, it should be emphasized that the CRC of a parametric polynomial on an interval contains more information than is needed for these problems, and consequently it has more potential applications than the examples given here.

2. PRELIMINARY

In this section, we review some definitions and theorems which mainly come from [10] and [6]. They are necessary for the new algorithm. Let $p(x) \in \mathbb{R}[x]$ with $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $a_n \neq 0$.

Definition 1. *The $2n \times 2n$ matrix M*

$$\begin{pmatrix}
a_n & a_{n-1} & a_{n-2} & \ldots & a_0 & & \\
0 & na_n & (n-1)a_{n-1} & \ldots & a_1 & & \\
 & a_n & a_{n-1} & \ldots & a_1 & a_0 & \\
 & 0 & na_n & \ldots & 2a_2 & a_1 & \\
 & & & \ldots & \ldots & & \\
 & & & \ldots & \ldots & & \\
 & & & a_n & a_{n-1} & \ldots & a_0 \\
 & & & 0 & na_n & \ldots & a_1
\end{pmatrix}$$

is called the discrimination matrix of p.

Definition 2. *For $1 \le k \le 2n$, let M_k be the kth principal minor of M, and let $D_k = M_{2k}$. The n-tuple $D = [D_1, D_2, \ldots, D_n]$ is called the discriminant sequence of p.*

Definition 3. *If $\operatorname{sgn} x$ is the signum function, $\operatorname{sgn} 0 = 0$, then the list $[s_1, s_2, \ldots, s_n] = [\operatorname{sgn} D_1, \operatorname{sgn} D_2, \ldots, \operatorname{sgn} D_n]$ is called the sign list of p.*

Definition 4. *The revised sign list $[e_1, e_2, \ldots, e_n]$ of $p(x)$ is constructed from the sign list $s = [s_1, s_2, \ldots, s_n]$ of p as follows. If $[s_i, s_{i+1}, \ldots, s_{i+j}]$ is a section of s, where $s_i \neq 0$, $s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0$ and $s_{i+j} \neq 0$, then we replace the subsection $[s_{i+1}, \ldots, s_{i+j-1}]$ by*

$$[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \ldots],$$

i.e., let $e_{i+r} = (-1)^{\lfloor (r+1)/2 \rfloor} s_i$, for $r = 1, 2, \ldots, j-1$, and keep other elements unchanged, i.e., let $e_k = s_k$. The revised sign list of p (resp. s) is denoted by $\operatorname{rsl}(p)$ (resp. $\operatorname{rsl}(s)$).

Yang, Hou and Zeng used the following theorem to calculate the number of distinct complex-conjugate roots and real roots.

Theorem 1. *Suppose a polynomial $p \in \mathbb{R}[x]$ has revised sign list $\operatorname{rsl}(p)$. If the number of non-vanishing members of $\operatorname{rsl}(p)$ is s, and the number of sign changes in $\operatorname{rsl}(p)$ is v, then $p(x)$ has v pairs of distinct complex-conjugate roots and $s - 2v$ distinct real roots.*

In order to calculate the multiplicities of roots, Yang, Hou and Zeng used the following definitions and propositions.

Definition 5. *Let $\Delta(p)$ denote $\gcd(p(x), p'(x))$, and let $\Delta^0(p) = p(x)$, $\Delta^j(p) = \Delta(\Delta^{j-1}(p))$, $j = 1, 2, \ldots$. Then $\Delta^0(p), \Delta^1(p), \Delta^2(p), \ldots$ is called the Δ-sequence of p.*

Proposition 1. *If $\operatorname{rsl}(p)$ contains k zeros, equivalently, $D_n = \cdots = D_{n-k+1} = 0$ but $D_{n-k} \neq 0$, then $\gcd(p, p') = P_k(p, p')$, where $P_k(p, p')$ is the kth subresultant of $p(x)$ and $p'(x)$.*

The relationship between the RC of $\Delta^j(p)$ and the RC of its 'repeated part' $\Delta^{j+1}(p)$ is given by the following propositions.

Proposition 2. *If $\Delta^j(p)$ has k distinct roots with respective multiplicities $n_1, n_2, \ldots, n_k$, then $\Delta^{j+1}(p)$ has at most k distinct roots with respective multiplicities $n_1 - 1, n_2 - 1, \ldots, n_k - 1$.*

Proposition 3. *If $\Delta^j(p)$ has k distinct roots with respective multiplicities $n_1, n_2, \ldots, n_k$, and $\Delta^{j-1}(p)$ has m distinct roots, then $m \ge k$, and the multiplicities of these m distinct roots are $n_1 + 1, n_2 + 1, \ldots, n_k + 1, 1, \ldots, 1$ respectively.*

However, the old algorithms [5] and the methods above have to work with revised sign list which is a major source of inefficiency, since we have to transfer the output conditions in terms of revised sign lists to conditions in terms of sign lists. The transferring process is usually very difficult and full of opportunities for including non-realizable conditions. This consideration motivated the authors to propose a new algorithm for overcoming these disadvantages [6]. The new algorithm offers improved efficiency and a new test for non-realizable conditions. The improvement lies in the direct use of sign lists, rather than revised sign lists.

The algorithm uses the following definitions and theorems, where "PmV" means "generalized Permanences minus Variations" [3].

Definition 6. *Let $s = [s_n, \ldots, s_0]$ be a finite list of elements in $\mathbb{R}$ such that $s_n \neq 0$. Let $m < n$ such that $s_{n-1} = \cdots = s_{m+1} = 0$, and $s_m \neq 0$, and $s' = [s_m, \ldots, s_0]$.*

If there is no such m, then s' is the empty list. We define inductively

$$\mathrm{PmV}(s) = \begin{cases} 0, & s' = \emptyset; \\ \mathrm{PmV}(s') + \epsilon_{n-m}\,\mathrm{sgn}(s_n s_m), & n-m \text{ odd}; \\ \mathrm{PmV}(s'), & n-m \text{ even}; \end{cases}$$

where $\epsilon_{n-m} = (-1)^{(n-m)(n-m-1)/2}$.

The following theorem gives the number of distinct roots in terms of sign lists.

Theorem 2. *Let $D = [D_1, \ldots, D_n]$ be the discriminant sequence of a real polynomial $p(x)$ of degree n, and ℓ be the maximal index such that $D_\ell \neq 0$. If $\mathrm{PmV}(D) = r$, then $p(x)$ has $r+1$ distinct real roots and $\frac{1}{2}(\ell - r - 1)$ pairs of distinct complex conjugate roots.*

The next theorem can be used to detect the non-realizable sign lists in output conditions.

Theorem 3. *Let $S = [s_1, \ldots, s_n]$ and $R = [r_1, \ldots, r_n]$ be the sign list and the revised sign list of $p(x)$ respectively. Then $\mathrm{PmV}(S) = \mathrm{PmV}(R)$.*

At last, we review a result given by Yang and Xia [9][11] for computing the number of real roots on intervals, which gives us some clue for solving the first problem mentioned in Section 1.

Let $p \in \mathbb{R}[x]$ with $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $a_n \neq 0$. Let $a, b \in \mathbb{R}$ such that $p(a) \neq 0$ and $p(b) \neq 0$. Let $\phi_1(x) = (1-x)^n p\left(\frac{b-ax}{1-x}\right)$ and $\phi_2(x) = \phi_1(-x^2) = (1+x^2)^n p\left(\frac{b+ax^2}{1+x^2}\right)$. Then, it is easy to see that $\mathrm{coeff}(\phi_1, x, n) = (-1)^n p(a) \neq 0$, $\mathrm{coeff}(\phi_2, x, 2n) = p(a) \neq 0$ and $\phi_1(0) = \mathrm{coeff}(\phi_1, x, 0) = \mathrm{coeff}(\phi_2, x, 0) = p(b) \neq 0$. Furthermore

Proposition 4. $\#\{x \in (a,b) \mid p(x) = 0\} = \#\{x < 0 \mid \phi_1(x) = 0\} = \frac{1}{2}\#\{x \in \mathbb{R} \mid \phi_2(x) = 0\}$.

3. BASIS OF THE ALGORITHM

In this section, we establish the basis for the new algorithm. The main idea is that we transfer the computation of CRC for a parametric polynomial on an interval to the computation of CRC for a parametric polynomial on $\mathbb{R}$.

Theorem 4. *Let $p(x), \phi_1(x), \phi_2(x)$ be defined as in Section 2. Then, $[r_1, r_2, \ldots, r_k]$ is the RC of $p(x)$ on (a, b), if and only if $[r_1, r_1, r_2, r_2, \ldots, r_k, r_k]$ is the real RC of $\phi_2(x)$ on $\mathbb{R}$.*

Proof. Since $[r_1, r_2, \ldots, r_k]$ is the RC of $p(x)$ on (a, b), we can decompose $p(x)$ in $\mathbb{C}$ as

$$p(x) = a_n \prod_{i=1}^{k}$$

PolySL.

Input: ª 2 and L_2.

Output: The set of all possible sign lists of ª 2.

Procedure:

- Compute the discriminant sequence $D = [D_1; \ldots ; D_{2n}]$ of ª 2.

- Compute the set S_0 of all possible sign lists from D: for $1 \cdot k \cdot 2n$, if D_k 2 R, then D_k ! sgn(D_k), otherwise, 2df (4(1)is(4(4Tf1.30433.18.0(4(4T)Tj433-25.94(4T5.63-0.99Td4(4Tj/T1_38.913.310Td(k)Tj[.96Tf6457T)Tj457TTd(n)Tj/ D_k ! f ¡ 1; 0; 1g. For example, if D = [1; ¡ 2; a], then S_0 = f[1; ¡ 1; ¡ 1]; [1; ¡ 1; 0]; [1; ¡ 1; 1]g.

- Compute S = fs 2 S_0j PmV(s) = PmV(rsl(s)) = 2k ¡ 1g b3/T1_38.96Tf8.70Td(D)Ln

IntCRC

Input: A real parametric polynomial $p(x)$, and $a; b$ 2 R [f ¡ 1 ; + 1 g.

Output: The CRC of $p(x)$ on $(a; b)$.

Procedure:

```
L Ã AllRC(deg(p))
compute ª_2
for L in L do
    C Ã IntCond(ª_2; DRC(L))
    if C 6= NULL then
        return L and C
```

Optimization of algorithm. Finally we discuss the optimization of the algorithm. In comparison with the case on R, the output conditions of the CRC of a parametric polynomial on an interval is usually large, especially when the parametric polynomial has a general form. So there remains the work of condensing the output conditions. Suppose $[D_1; : : : ; D_{2n}]$ is the discriminant sequence of ª$_2$, and S is the set of all possible sign lists of ª$_2$ for ª$_2$ having L_2 = [

f ¡ 1

All possible sign lists of P_6 would be $[1,1,1,-1,0,0]$, $[1,0,0,-1,0,0]$,$[1,-1,-1,-1,0,0]$, $[1,1,1,-1,1,1]$, $[1,-1,0,0,1,1]$, $[1,0-,-1,-1,1,1]$,$[1,1,1,<>,-1,1]$, $[1,1,1,0,0,1]$, $[1,-1,-1,1,1,1]$, $[1,-1,-1,0,0,1]$, $[1,-1,-1,-1,-1,1]$, $[1,0,0,-1,<>,1]$.

Now these sign lists of P_6 can be divided into two groups: $G_4 = \{[1,1,1,-1,0,0],[1,0,0,-1,0,0],[1,-1,-1,-1,0,0]\}$ and $G_6 = \{[1,1,1,-1,1,1],[1,-1,0,0,1,1],[1,0-,-1,-1,1,1],[1,1,1,<>,-1,1],[1,1,1,0,0,1],[1,-1,-1,1,1,1]$, $[1,-1,-1,0,0,1],[1,-1,-1,-1,-1,1],[1,0,0,-1,<>,1]\}$.

If the sign list of P_6 belongs to G_4, then the number of distinct roots of P_6 is 4. So the 'repeated part' $\Delta^1(P_6) = P_{62}$ and the RC of P_{62} is MinusOne($[1,1]$) = [ ]. For P_{62} and [ ], IntCond is called again, obtaining that the condition for P_{62} having [ ] as its real RC on R is its sign list being $[1,-1]$.

At this point, the termination condition 2 is satisfied, so IntCond terminates. If the sign list of P_6 belongs to G_6, then the termination condition 3 is satisfied, and IntCond terminates.

In summary, p_3 has [1] as its RC on $(0,2)$, if and only if the sign list of P_6 belongs to G_4 and the sign list of P_{62} is $[1,-1]$, or the sign list of P_6 belongs to G_6. The cases $[\,],[2]$ and $[1,1]$ can be explained similarly.

For the cases $[3],[1,2],[1,1,1]$, since the output of IntCond is the empty sequence NULL, they are not realizable. Based on the CRC of p_3, we can answer some questions concerning real quantifier elimination. The discriminant sequence of P_6 is $[1,D_2,D_3,D_4,D_5,D_6]$, where

$$
\begin{aligned}
D_2 &= -3b^2 - 2ab,\\
D_3 &= -a^2b(2a+3b),\\
D_4 &= a^2b(a^2b+9b^2+2a^3+6ab),\\
D_5 &= -b(a^2b+2a^3+6ab+9b^2)(4a^3+27b^2),\\
D_6 &= -(8+2a+b)b(4a^3+27b^2)^2.
\end{aligned}
$$

The necessary and sufficient condition for $\forall x \in (0,2)[p_3 \neq 0]$ is that case (1) holds, and case (1) holds iff the sign list of P_6 be one of the following: $[1,-1,0,0,<>,-1]$, $[1,-1,-1,1,0,0],[1,-1,-1,0+,0,-1],[1,-1,-1,1,-1,-1]$, $[1,-1,-1,*,1,-1],[1,0,0,-1,1,-1],[1,1,1,-1,1,-1]$. Therefore, the necessary and sufficient condition for $\forall x \in (0,2)[p_3 \neq 0]$ is

$[D_2 < 0 \wedge D_3 = 0 \wedge D_4 = 0 \wedge D_5 \neq 0 \wedge D_6 < 0] \vee [D_2 < 0 \wedge D_3 < 0 \wedge D_4 > 0 \wedge D_5 = 0 \wedge D_6 = 0] \vee [D_2 < 0 \wedge D_3 < 0 \wedge D_4 \geq 0 \wedge D_5 = 0 \wedge D_6 < 0] \vee [D_2 < 0 \wedge D_3 < 0 \wedge D_4 > 0 \wedge D_5 < 0 \wedge D_6 < 0] \vee [D_2 < 0 \wedge D_3 < 0 \wedge D_5 > 0 \wedge D_6 < 0] \vee [D_2 = 0 \wedge D_3 = 0 \wedge D_4 < 0 \wedge D_5 > 0 \wedge D_6 < 0] \vee [D_2 > 0 \wedge D_3 > 0 \wedge D_4 < 0 \wedge D_5$ ≠ 0 ∧ ∧ and _ 189 (su±cien) 8.9t _ 18.9condition _ 18.9for

```
[P10,[1,0,0,0,0,1,1,1,0,0]],[P102,[1,-1]];[P10,
 [1,0,0,0,0,1,1,1,1,-1],[1,0,0,0,0,1,0,0,-1,-1],
 [1,0,0,0,0,1,1,*,-1,-1],[1,0,0,0,0,1,1,0+,0,-1],
[1,0,0,0,0,0+,0,-1,-1,-1],[1,0,0,0,0,*,-1,-1,-1,-1]]

(7) [1,1,1], if and only if
[P10,[1,0,0,0,0,1,1,1,1,1]]
Where,
(#1) P1042:=-2*b*x^2-5*c
(#2) P102:=54*a^4*c+27*b*a^4*x^2-225*x^2*c^2*a^2
+600*a*c^2*b+720*a*x^2*c*b^2-320*c*b^3-256*x^2*b^4,
(#3) P10:=x^10+a*x^4+b*x^2+c,
(#4) P104:=-3*a*x^4-4*b*x^2-5*c,
 and the initial condition is
 c <> 0
```

The discriminant sequence of P_{10} is $[1; 0; 0; 0; 0; D_6; D_7; D_8; D_9; D_{10}]$, where

$$
\begin{aligned}
D_6 &= ¡ a^5; \qquad D_7 = ¡ a^3(27a^4 + 300abc ¡ 160b^3);\\
D_8 &= (300bac ¡ 160b^3 + 27a^4)(720acb^2 ¡ 256b^4 + 27a^4b\\
&\quad ¡ 225a^2c^2);\\
D_9 &= ¡ (720acb^2 ¡ 256b^4 + 27a^4b ¡ 225a^2c^2)(¡ 1600b^3ca\\
&\quad + 256b^5 ¡ 27a^4b^2 + 2250ba^2c^2 + 3125c^4 + 108a^5c);\\
D_{10} &= ¡ c(¡ 1600b^3ca + 256b^5 ¡ 27a^4b^2 + 2250ba^2c^2\\
&\quad + 3125c^4 + 108a^5c)^2
\end{aligned}
$$

Again, we assume that the initial condition $c \neq 0$ holds. Then $(8 x > 0)[p_5 = x^5 + ax^2 + bx + c > 0]$ i® case (1) holds. That is $[D_6 < 0 \wedge D_7$

D_7 =1Td (2) Tj/T1_ 8f596 0 -174.723Tf5.233.81Td (5) Tjv + n. .040Td1a holds.