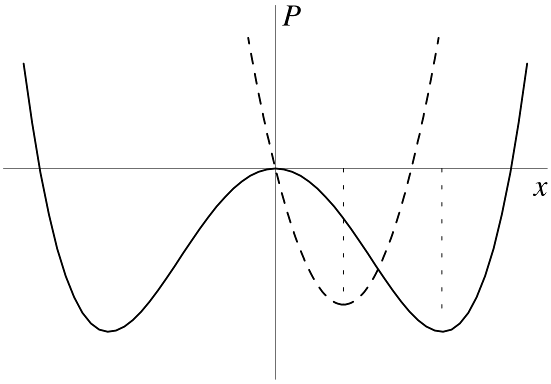
has to be stretched in order to accommodate algorithms. Maple does this when the command `minimize`$(P(x), x)$ returns

$$\frac{1}{16a_4}\left[(8a_4a_2 - 3a_3^2)X^2 + (12a_4a_1 - 2a_3a_2)X - a_3a_1\right] + a_0 \ ,$$


P
x

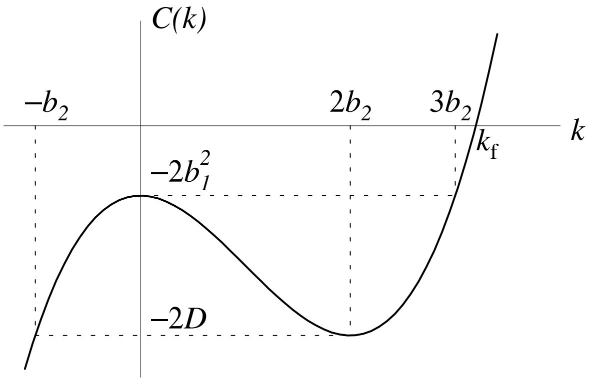


Figure 2: A graph of the auxiliary cubic for $b_2 > 0$. It can be seen that the root $k_{\rm f}$ is positive and greater than $3b_2$ as required. The quantity $D = b_1^2$

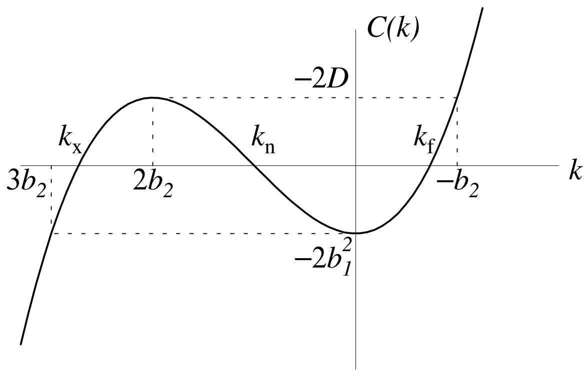


Figure 4: A graph of the auxiliary cubic for $b_2 < 0$. The quantity $D = b_1^2 + 2b_2^3$ is negative, so there are 3 real roots. They are marked on the figure as k_{x}, k_{n} and k_{f}.

Denote the positive solution of (9) by k_{f}. Since $k_{\mathrm{f}} > 3b_2$, the minimum of (4) is obtained from (8) as

$$\inf(P_4) = -b_1^2/k_{\mathrm{f}} - \tfrac{1}{4}(k_{\mathrm{f}} - 3b_2)^2 \ .$$

The first term of this expression is better transformed so that b_1 does not appear explicitly, for reasons that will be given below. The transformation is made by rewriting (9) in the form

$$\tfrac{1}{2}k^2 - \tfrac{3}{2}b_2 k = b_1^2/k \ , \tag{10}$$

and hence (5) is obtained.

It remains to find an explicit formula for k_{f}. The expression (6) [illegible] st[illegible]dard solution of (9), but the fact that it gives the positive solution of (9) must be ve[illegible]ie[illegible] Rewrite (7), introducing D, as $s = b_1^2 + b_2^3 + \sqrt{b_1^2 D}$. First, consider the case $b_2 \geq 0$ an[illegible] ≥ 0; all terms in (6) are real and positive. Second, consider $b_2 <$ [illegible] D [illegible] 3

Proof: If $b_2 \geq 0$, then clearly the minimum is 0 when $y = 0$. If $b_2 < 0$, then completing the square can be used again to give the minimum as $-\frac{9}{4}b_2^2$ at $y^2 = -\frac{3}{2}b_2$. The theorem uses the minimum function to combine these cases.

Now there is an interesting development. One takes formula (5) for $M(b_1, b_2)$, forgets that it was derived for $b_1 \neq 0$, and substitutes $b_1 = 0$. For the case $b_2 > 0$, one computes $k_{\mathrm{f}} = 3b_2$ and $M(0, b_2) = 0$. For $b_2 < 0$, equation (11) can be reused with $\theta = \pi$, making $s = -b_2$ and $k_{\mathrm{f}} = 0$. Then $M(0, b_2) = -\frac{9}{4}b_2^2$. Thus for these cases, M continues to give the correct result. This is because of the transformation (10). For $b_2 = 0$, k_{f} contains a term $0/0$ and this prevents a simple substitution from obtaining $M(0, 0) = 0$, or in other words, $M(b_1, b_2$

Theorem 6 *With the notation already defined, P_4 has a relative maximum when $D < 0$ and k_x is the root of (k) satisfying $3b_2 < k_x < 2b_2$. The formula for k_x is*

$$k_x = s^{1/3}e^{2\pi i/3} + b_2^2 s^{-1/3}e^{-2\pi i/3} + b_2 \ .$$

Properties of the solutions

In spy stories, everyone seems to think that possessing the formula is all that is required — a bit like a weak student facing a mathematics exam. However, one must be able to use it. The formulae just derived can be used to show that the turning points of P_4 have some interesting properties. For example, the sign of b_1 is all that decides the side of the origin on which the infimum lies; if there is a secondary minimum, then both it and the local maximum are always on the same side of the origin, and that is the opposite side from the infimum; the infimum is always further away from the origin than the secondary minimum.

After several pages of algebra, it is always comforting to try a few numerical examples and see that everything works out. No one wants a repeat of the last scenes of *Th Malt s Falcon*. In addition, the examples here carry some useful lessons of their own. So consider $y^4 - 14y^2 - 24y$, which of course has been carefully rigged to have integer turning points. Substituting $b_2 = -14/3$ and $b_1 = -12$ into (6) and asking Maple to simplify the result gives

$$k_f = \frac{2\left(143 + 180\sqrt{3}i\right)^{2/3} + 98 - 14\left(143 + 180\sqrt{3}i\right)^{1/3}}{3\left(143 + 180\sqrt{3}i\right)^{1/3}} \ .$$

All computer systems can approximate this to 4.0000000, but none can automatically simplify it to the exact number 4. This is an unavoidable difficulty associated with solving a cubic using the standard formulae. The simplification $\left(143 + 180\sqrt{3}\,i\right)^{1/3} = \frac{1}{2}(13 + 3\sqrt{3}\,i)$, which is needed to obtain the exact result, is not implemented in any present computer system; perhaps not many humans would make the simplification spontaneously either. Of course most of the time, no simplification is possible. In any event, the infimum is at $x_f = 3$, and equals -117. In the same way, and with the same difficulties, $k_n = -6$ and $k_x = -12$.

Simply using the formula M given in (5) to compute a numerical minimum is not a very interesting application. A more challenging question is to find the values of p that make the polynomial $x^4 + 3px^2 + 2x + 2$ positive for all x. This type of problem is a simple example of quantifier elimination [1]. The condition is simply $M(1, p) + 2 > 0$, which becomes a long messy inequality when written out explicitly. Plotting the expression numerically shows that the answer is $p > -1/3$, but an analytic proof is a real challenge.

The final example does not aspire to present a general method for a class of problems, but the following challenge arose at the time of writing this paper. Given the points (x_i, y_i) equal to $(0, 4)$, $(1, 2)$, $(3, 1)$, $(4, 2)$, $(6, 5)$, find a convex polynomial that passes through them. The Lagrange interpolating polynomial is a quartic:

$$y_L =)$$

For all a and b, this passes through the given points. Two derivatives of this give a quartic inequality $y_S'' > 0$, which will be satisfied if $\inf y_S'' > 0$. This reduces to an inequality in the two variables a and b after using (5). Plotting contours shows that a region exists that satisfies the constraint, and in particular it includes the rectangle $1/2000 < a \leq 1/1000$, $0 < b \leq 1/1000$.

A computer epilogue

Just when you think it is time to roll the references, a familiar character reappears: the computer. The introduction mentioned that many CAS have routines to minimize functions, so the programming aspects of Theorem 3 are of interest. It was seen above that if (6) is used with numerical coefficients, the system might obtain a result that is correct, but not in the simplest form. In a similar way, if the evaluation of (6) generates intermediate quantities that are complex, a small nonzero imaginary part can appear in the final result because of rounding errors.

An alternative to the explicit formula (6) is to use a token such as the `RootOf` offered in Maple V release 4. To the description above, we can add that it also accepts a third argument, in the form of an interval that brackets the required root. The interval could be specified using (6), of course, but it is better to surrender precision to gain algebraic simplicity. Now $k_f \to 3b_2$ as $b_2 \to \infty$, and $k_f \to (2b_1^2)^{1/3}$ as $b_2 \to 0$, and $k_f \to \sqrt{-2b_1^2/3b_2}$ as $b_2 \to -\infty$. An estimate that takes these limits into account, while staying with integral powers is $k_a = 1 + 3|b_2| + \frac{2}{3}b_1^2$, which is an upper bound on k_f because $(k_a) > 0$. Thus, the expression (6) can be replaced by

$$k_f = \mathtt{RootOf}(k^3 - 3b_2k^2 - 2b_1^2, k, 0 \,..\, 1 + 3|b_2| + \tfrac{2}{3}b_1^2) \;,$$

where some artistic licence has been taken with Maple's input language. Similar constructions exist in other systems. The advantages of this approach are that the system has the possibility of obtaining the best representation of the root directly, and that the case $b_1 = b_2 = 0$ is no longer a removable singularity. The disadvantages are that the representation is unfamiliar, and it may not allow the further analysis that is possible with the explicit form; also the simplification routines existing for this type of construction are not yet at all strong.

A final comment repeats what has been achieved from a slightly different perspective. The paper opened with a cubic equation (2) whose roots gave turning points. The main theorem replaced this with a different cubic. One cannot avoid solving a cubic sooner or later, but the auxiliary cubic in (9) has the advantage that one knows in advance which root to select and where it will be.

Acknowledgement. The last example came from a discussion with S. Watt, J. Grimm and A. Galligo; in particular, I thank J. Grimm for showing me some related results obtained from a more general point of view. This article was written while on leave at INRIA, Sophia Antipolis, France, and the hospitality of S.M. Watt is gratefully acknowledged.

One of the referees proposed a counterexample to theorem 1: unfortunately my agents have been unable to locate the movie *Sneakers* to verify this intelligence report.

References

[1] Lazard, D. Quantifier elimination: optimal solution for two classical problems, *J. Symbolic Comp.* 5 (1988), 261–266.