The multi-valued nature of inverse functions

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Abstract

A new treatment is given of the elementary inverse functions. The new approach addresses the difference between the single-valued inverse function defined by computer systems and the multi-valued function which represents the multiple solutions of the defining equation. The approach takes an idea from complex analysis, namely the *branch* of an inverse function, and defines an index for each branch. The branch index then becomes an additional argument to the (new) function. A benefit of the new approach is that it helps with the general problem of correctly simplifying expressions containing inverse functions, which has always been difficult both for humans and for computer algebra systems. The new approach also can be extended to non-elementary inverse functions such as the Lambert W function, which otherwise cannot be handled. The difference between this approach and that of Riemann surfaces lies in the fact that Riemann surfaces distinguish between branches by dividing the *domain* of the function into sheets, whereas here the range of the function is indexed.

1 Introduction

Two developments in mathematics suggest the need for a new treatment of multi-valued functions, including the elementary inverse functions. The developments are, first, the implementation of inverse functions in computer-based mathematical systems and, second, the appearance in the literature of new inverse functions. The computer systems have struggled for years to find the best way to handle possible simplifications such as

$$(z^n)^{1/n} \,?=?\, z \;, \quad \arcsin(\sin z) \,?=?\, z \;, \quad \ln(e^z) \,?=?\, z \;, \tag{1}$$

as indeed have mathematicians [5, 7]. Numerical counter-examples are

$$((-1)^2)^{1/2} \neq -1 \;, \quad \arcsin(\sin 2\pi) \neq 2\pi \;, \quad \ln(e^{3\pi i}) \neq 3\pi i \;.$$

In the 1980s, mistakes like these could commonly be found in computer algebra systems[1]. The new treatment offers one way of looking at such problems. The other motivation is the study of the Lambert W [6] and other inverse functions, which have no trivial relation between their branches, in contrast to the elementary inverse functions.

There are, in addition, æsthetic reasons. Anyone who has taught inverse trigonometric functions, or the complex roots of a number, knows how difficult students find the idea of multi-valued functions. One of the reasons is that there is not really a single uniform treatment. For example, every calculus textbook introduces inverse functions with a discussion of multi-valuedness and then ignores it when justifying equations such as

$$\int \frac{dx}{1+x^2} = \arctan x$$

(Abramowitz & Stegun [1] do this in the same chapter). Of course there will always be different treatments of the subject, because of the mathematical desire for a different point of view. Mathematical topics are to mathematicians rather like antique vases are to vase connoisseurs. The connoisseurs are not content to look at their vases only from the front; they want to pick them up and admire them from all angles. In the same way, the mathematical pleasure of a topic is not exhausted by any single treatment, however thorough. Perhaps there is something of this in the present treatment, but it is argued that there are practical reasons to change, and practical benefits to gain.

[1] Let's not point fingers at particular systems.

2 A question of values

The

century scientist J. H. Lambert that

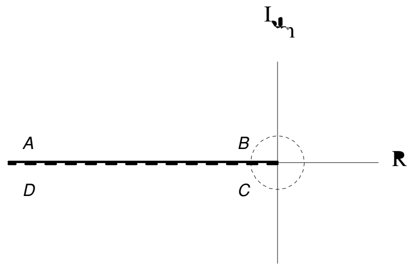


Figure 1: The z-plane labelled with branch cut and points for mapping to the p-plane.

sometimes

Branch 1: $p = \ln_1 z$

Branch 0: $p = \ln z$

Branch $-$

6.3 Cosine

Since $\sin(p - \pi/2) = -\cos(p)$ it is obvious that the inverse function will have a similar branch rule to invsin. In order to ensure the principal branch is branch 0 and har

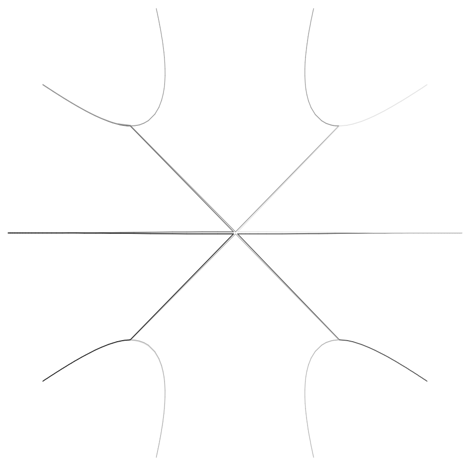
7.1 Composition

Let f be a single-valued function, for example one of those listed in the last section, and let $\check{f}_k$ be its (set of) inverse functions. It is well known that $f(\check{f}_k(z)) = z$ for all z and k, but $\check{f}_k(f(z)) \neq z$ except when z lies in a certain domain. Let the range of $\check{f}_k$ in the complex plane be $\mathbb{C}_k \subset \mathbb{C}$. Then $\check{f}_k(f(z)) = z$ provided $z \in \mathbb{C}_k$. In this notation, the vague statement $\mathrm{Arcsin}(\sin z) = z$ can be made precise in two ways. The simple way is to write $\exists k, \mathrm{invsin}_k \sin z = z$; the other way is to say what k is.

For the elementary functions, it is possible to write down a rule for $\check{f}_k(f(z))$ for any z, using the unwinding number $\mathcal{K}(z) = \left\lceil \frac{z-\pi}{2\pi} \right\rceil$, defined in [5] (rather than in [7] where the sign is different). For example, the equations in (1) become

$$\begin{aligned}
[z^n]_k^{1/n} &= z e^{2\pi i(\mathcal{K}(n \ln z)+k)/n} = z C_n(z) e^{2\pi i k/n} \,, \\
\mathrm{invsin}_k(\sin z) &= z(-1)^{k+\mathcal{K}(2iz)} - \pi \left((-1)^{k+\mathcal{K}(2iz)}\mathcal{K}(2iz) - k \right. \,, \\
\ln_k e^z &= z + 2\pi i(\mathcal{K}(z) + k) \,, \\
\mathrm{invtan}_k(\tan z) &= z + \pi(k - \mathcal{K}(2iz))
\end{aligned}$$

For any value of z, there is a value of k which reduces the composition to the identity. The factor $C_n(z)$ above is a generalization of the function `csgn(z)` that regularly



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