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Abstract

We consider, from a symbolic point of view, a pair of definite integrals containing Lambert , recently considered from a numerical point of view by Walter Gautschi. We transform the integrals to a shape that can be integrated in special cases by a computer-algebra system or by using tables of integrals, such as Prudnikov .

1 Introduction

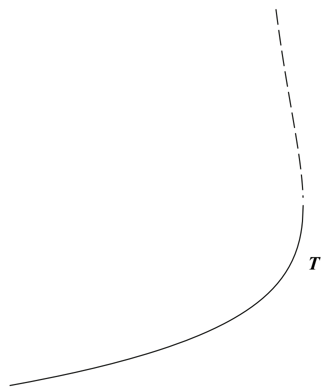
The paper [4] examines, in e ect, numerical schemes for the evaluation of the integrals[1]

$$I_0(,) = \int_1^{\infty} {}_0 x \quad -x^{\alpha} \quad x^{-\beta} \quad x \tag{1}$$

and

$$I_1(,) = \int_0^{1} {}_1 x \quad -x^{\alpha} \quad x^{-\beta} \quad x \;, \tag{2}$$

where and are restricted to values ensuring convergence[2]. Here, $_k$ is the Tree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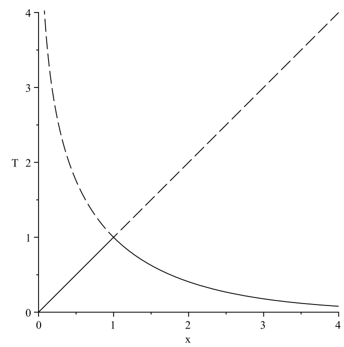

Figure 2: The equation $y\exp(-y) = x\exp(-x)$ has two solutions for y, which we may write as $y = {}_k(x\exp(-x))$ for $= 0$ (solid line) and $= 1$ (dashed line). Notice that each curve has a corner at $x = 1$; at that point the trivial solution $y = x$ crosses the nontrivial solution, and the descriptions of each solution in terms of the Tree function changes. A parametric description of the nontrivial solution is given by $y = v\exp(v)/(\exp(v)-1)$ and $x = v/(\exp(v)-1)$ where v runs from $-\infty$ to ∞. If $v \geq 0$, we recover the branch with $x \leq 1$, whereas if $v \leq 0$ then $x \geq 1$.

2 Branch Di erences

$$I_0(2,3) = \int_0^\infty \frac{e^{-2v}(1-(1+v)e^{-v})}{v\,(1-e^{-v})}\, v = \tfrac{3}{2} - \quad - \ln 2\,, \tag{13}$$

and

$$I_0(3,2) = \int_0^\infty \frac{e^{-3v}\,v\,(1-(1+v)e^{-v})}{(1-e^{-v})^3}\, v = \tfrac{5}{2} - 6\zeta(3) + \tfrac{1}{2}\,{}^2\,. \tag{14}$$

Similarly, I_1 is easily evaluated in Maple if and are integer values (or are equal).

$$I_1(2,0) = -\int_0^\infty \frac{v^2 e^{-v}(1-v-e^{-v})}{(1-e^{-v})^4}\, v = \tfrac{1}{3} + 2\zeta(3) + \tfrac{1}{3}\,{}^2\,. \tag{15}$$

An interesting variation is that Maple can sometimes evaluate the integrals if and di er by an integer but are not themselves integers:

$$I_0\left(\tfrac{3}{2},\tfrac{1}{2}\right) = \int_0^\infty {}^{-3v/2}\frac{(1 - {}^{-v} - {}^{-v})}{(1 - {}^{-v})^3}\, v = -\tfrac{1}{2} + \tfrac{3}{4}\,{}^2 - \tfrac{21}{4}\zeta(3)\,. \tag{16}$$

This does not always work, however. If = 1/4 and = −3/4, then the di erence is an integer but neither Maple nor Mathematica is able to evaluate the integral. For the simpler integral $I = I_0(\ ,\ ) + I_1(1-\ ,1-\ )$ this also happens.

At the time of writing, we do not know if any computer algebra system can evaluate these integrals for values of and that have non-integer di erences, or even merely for arbitrary and whose di erence is an integer.

4 Series

We pointed out earlier that when = the integrals for I_0 and I_1 could be identified as containing Ψ_1, the trigamma function, by using a series expansion. One is tempted fo.27l6(I)Tj /F6 1 Tf3 Tf 7(Ψ)-339.9(is)-340.9(te

When = this reduces, as claimed, to

$$\Psi_1(\ ) = \sum_{\geq 0} \frac{1}{(\ +\ )^2}\,. \tag{20}$$

If we introduce a new variable m with the definition $= \ -m$, the sum in (19) becomes

$$(m,\ ) = \sum_{\geq 0} \binom{\ m+\ }{\ } \frac{1}{(\ +\ )^{m+2}}\,. \tag{21}$$

For explicit integers m, Maple can evaluate this sum in terms of known special functions such as the polygamma functions $\Psi_j(\ )$ for $j \leq m+1$. For example,

$$(4,\ ) = \frac{1}{24}\Psi_1(\ ) + \left(\frac{1}{12}\ -\frac{5}{24}\right)\Psi_2(\ ) + \left(-\frac{5}{24}\ +\frac{1}{24}\right)$$

5 Comparison with Examples from [4]

We have seen here a reference solution for some examples in which and di er by an integer. For those integrals for which we cannot derive a symbolic solution, we could use numerical methods such as those in Maple's /I , but this is less interesting. Tables 1 and 2 show that Gautschi's results are as accurate as he claimed.

Numerical integration of I_0 and I_1 in the forms containing is not challenging in Maple; Gautschi's task was to do it outside a system that had easy and accurate evaluation of , or . Numerical evaluation of the exponential forms given here is also not challenging: the singularities at $v = 0$ are removable and standard tricks for the accurate evaluation of $e^{v} - 1$, or $\ln(1 - y)$ in an equivalent logarithmic form of the integral, make it easy. Moreover, Maple's /I is more powerful yet. It uses singularity detection and generalized series to eliminate most di culties [5], and has no trouble here.

6 Concluding Remarks

One aim of this paper is to provide reference expressions for the integrals (1) and (2) in terms of quantities such as , the Euler-Mascheroni Constant, and evaluations of functions such as the Riemann ζ function, which we consider to be known and partially understood.[4]

Since the discovery of these special forms was the result of examining the properties of the two real branches of the Tree function, as well as the properties of their di erence, the exploration of branch relations in other Lambert integrals may lead to further development of solutions to special cases.

This paper has shown a transformation that takes some integrals from forms containing , or , to an

Table 1: The relative error of Gautschi's approximations $G(\;,\;)$ when compared with exact symbolic values $I_0^{(s)}(\;,\;)$. Note that a dash indicates integrals for which we do not have symbolic expressions. We also include the di erences between Gautschi's approximations and Maple's /I using : , denoted $I_0^{(m)}$.

		$\left\lvert \frac{I_0^{( )}(\alpha,\beta)-G(\alpha,\beta)}{G(\alpha,\beta)} \right\rvert$	$\left\lvert \frac{I_0^{( )}(\alpha,\beta)-G(\alpha,\beta)}{G(\alpha,\beta)} \right\rvert$
2	2	3.58×10^{-32}	3.60×10^{-32}
	0	2.04×10^{-31}	2.04×10^{-31}
	-2	1.40×10^{-32}	1.40×10^{-32}
1	1	7.46×10^{-33}	7.44×10^{-33}
	0	2.20×10^{-32}	2.20×10^{-32}
	-1	2.31×10^{-32}	2.31×10^{-32}
$\frac{1}{2}$	2	-	3.35×10^{-33}
	0	-	2.24×10^{-32}
	-2	-	6.43×10^{-33}

Table 2: The relative error of Gautschi's approximations $G(\;,\;)$ when compared with exact symbolic values for *I*F4886Tj /F15 1 Tf3

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