2. Real solutions and rational solutions. In (1), the case $b<1$ can be avoided by the transformation $x \to -x$, so only the case $b>1$ is considered.

Theorem 1. Let $b, c \in \mathbb{R}$, $b>1$ and $n \in \mathbb{Z}$. Let $\sqrt[n]{c}$ denote the real branch of $c^{1/n}$. The solutions $x \in \mathbb{R}$ of the equation $x^n b^x = c$ are as follo s.

1. If n is odd and $c > -(n/e\ln b)^n$ or if n is even and $c \geq 0$,
$$x = \frac{n}{\ln b} W_0\left(\frac{\ln b}{n}\sqrt[n]{c}\right) .$$
2. If n is odd and $0 > c > -(n/e\ln b)^n$,
$$x = \frac{n}{\ln b} W_{-1}\left(\frac{\ln b}{n}\sqrt[n]{c}\right) .$$
3. If n is even and $0 < c \leq (n/e\ln b)^n$,
$$x = \frac{n}{\ln b} W_0\left(-\frac{\ln b}{n}\sqrt[n]{c}\right) , \quad \text{or} \quad x = \frac{n}{\ln b} W_{-1}\left(-\frac{\ln b}{n}\sqrt[n]{c}\right) .$$

Proof: Consider the case for n odd. Take the real-branch nth root of both sides of (1). Then it becomes, because n is odd,
$$xb^{x/n} = xe^{(x/n)\ln b} = \sqrt[n]{c} ,$$
and multiplying through by $(\ln b)/n$ and comparing ith (2) establishes that
$$x = \frac{n}{\ln b} W_k\left(\frac{\ln b}{n}\sqrt[n]{c}\right) .$$
Only the branches $k=0$ and $k=-1$ of W take real values [1]. The principal branch $W_0(x)$ is real for $x \geq -1/e$, hile $W_{-1}(x)$ is real for $-1/e < x < 0$. Imposing these restrictions leads to the cases listed in the theorem. The other cases are similar. It is clear that the converse statements are also true.

Example. The solutions of $x^2(9/16)^x = 3/16$ are
$$\frac{-2}{\ln(16/9)} W_0\left(\frac{\sqrt{3}}{8}\ln\frac{16}{9}\right) \approx -.387351650 ,$$
$$\frac{-2}{\ln(16/9)} W_0\left(-\frac{\sqrt{3}}{8}\ln\frac{16}{9}\right) = \frac{1}{2} ,$$
$$\frac{-2}{\ln(16/9)} W_{-1}\left(-\frac{\sqrt{3}}{8}\ln\frac{16}{9}\right) \approx 11.35508246 .$$
One must be a are of the possibility of equivalent expressions:
$$\frac{-2}{\ln(16/9)} W_0\left(-\frac{\sqrt{3}}{8}\ln\frac{16}{9}\right) = \frac{-1}{\ln(4/3)} W_0(-\frac{\sqrt{3}}{4}\ln\frac{4}{3}) = \frac{1}{2} .$$

We are led by the above to seek an algorithm for deciding hether $W_k(r_1$

Proof: By theorem 1, x as defined in (4) is a solution of

$$x^n(\mathrm{ipf}\, b)^x = c = r_1^n n^n (\mathrm{ipc}\, b)^n r_2 \ . \tag{6}$$

Then $x \in \mathbb{Q} \Rightarrow x \in \mathbb{Z}$, because if $x = p/q$, ith $p, q \in \mathbb{Z}$, then

$$(\mathrm{ipf}\, b^p)^{1/q} = c(q/p)^n \ .$$

Clearly the right-hand side is rational, but by the construction of ipf b, the left side is not in $\mathbb{Q}$ unless $q = 1$.

Computational algorithm. A practical procedure is as follo s. The quantity x defined in (4) is evaluated in t o stages. At first, the order of magnitude of x is determined; let it be $O(10^p)$. The evaluation is then repeated using more digits of precision than p, so that if x is indeed an integer, its value ill be determined correctly.[1] The proposed integral value is then accepted if it satisfies (5), other ise there is no rational simplification. If a computer system verifies (5) by direct computation of the left-hand side, very large integers are generated. It ould therefore be ise for a system to use the la s of exponents first.

Example. Returning to problem (3), one no computes

$$\frac{10561}{\ln(3/2)} W_0\left(\frac{528309}{168976}\ln\left(\frac{9}{4}\right)3^{\frac{7451}{10561}}2^{\frac{3110}{10561}}\right) \approx 39135.$$

 here 5 digits of precision have been used. The size of the result sho s that a computation accurate to 6 digits is sufficient (less than half the 16 needed by the continued fraction method), and this gives 39134.0 as the result. We no must verify that

$$(3/2)^{39134} - \left(\frac{528309}{168976}(10561)2\right.$$

and the symmetric solution (y, x). This implies simplification rules for W. We find from the $y = x$ solution, setting $y = 1/t$,

$$\begin{aligned} W_0\,(t \ln t) &= \ln t\ , \quad \text{for} \quad t > 1/e\ , \\ W_{-1}\,(t \ln t) &= \ln t\ , \quad \text{for} \quad 0 < t < 1/e\ . \end{aligned}$$

From the parametric solution, e obtain the additional relations

$$\begin{aligned} W_0\left(-\frac{\ln(1+s)}{s(1+s)^{1/s}}\right) &= -\frac{1+s}{s}\ln(1+s) \quad \text{for } -1 < s < 0\ , \\ W_{-1}\left(-\frac{\ln(1+s)}{s(1+s)^{1/s}}\right) &= -\frac{1+s}{s}\ln(1+s) \quad \text{for } 0 < s\ . \end{aligned}$$

These results are equivalent to a result by Lau erier [3, 4]. Equations (54–57) in [3] are obtained by riting $s = e^{2\Delta} - 1$. As of this riting, there are no computer algebra implementations of these (domain-specific) simplification rules.

4. The complex case. If **cas e.** If **cas e.** Ih

Example. For the equation $x^{1/3}256^x = 1$, the exponential factor in (8) is 1, but even so, W_0 and W_3 give solutions, but W_{-1}, W_1, W_2 do not, unless $x^{1/3}$ takes values from its other branches.

References

[1] Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J., and Knuth, D. E. On the Lambert W function. *Advances in* OI E I E V*in* OI E C Y