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On the Lambert W Function

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Abstract

L W fi $w \mapsto we^w$. I fl . W, ffi . W.

1. Introduction

I 1758, L $x = q + x^m$ x q. L [48,49]. I [28], E L

$$x \quad x = (\quad)vx^{\ +} \tag{1:1}$$

L x^- x $m =$ $q = (\quad)v$. E L

$$\begin{aligned} x^n = 1 + nv &+ \tfrac{1}{2}n(n + \quad + \quad)v^2 \\ &+ \tfrac{1}{6}n(n + \quad + 2 \quad)(n + 2 \quad + \quad)v^3 \\ &+ \tfrac{1}{24}n(n + \quad + 3 \quad)(n + 2 \quad + 2 \quad)(n + 3 \quad + \quad)v^4 \\ &+ \quad : \end{aligned} \tag{1:2}$$

A , E , = . (1.1) ( ) !

$$x = vx \quad : \tag{1:3}$$

E (1.3) $= 1$,

$\neq 0$. (1.3) , x x , $z = x$
$u =$ v. $z = uz$, (1.3) $= 1$.
(1.2), E fi $=$ $= 1$ (1.2)
$(x^n$ $1)=n$. N $n = 0$ x
:

$$x = v + \frac{2^1}{2!}v^2 + \frac{3^2}{3!}v^3 + \frac{4^3}{4!}v^4 + \frac{5^4}{5!}v^5 + \quad : \qquad (1:4)$$

, $jvj < 1=e$, fi $T(v)$
tree function [41]. I $W($ $v)$, $W(z)$ fi

$$W(z)e^{W(z)} = z \; : \qquad (1:5)$$

W T, W.
: ,
[10, 13, 25, 41, 67]; [64];
R´ [56, P III.209, . 146].
W, , -
ffi [69]. I [30], F , C
fi - $W(x)$ $x > 0$.
M -
W , 2
[14].

, ,
, W . -
W , B [11, . 27 28],
fi 0. ffi
(M 2) -
fl $W(z)$. F , K. B.
C F M ([58])
,
W.

notation

W , M .
Lambert W function, v

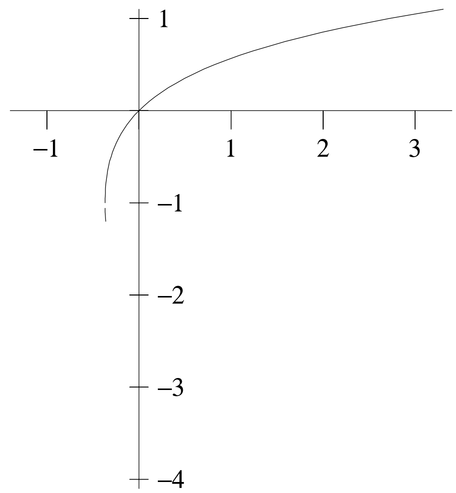

Figure 1. $W(x)$. , $W_0(x)$; , $W_{-1}(x)$.

$W(x)$ 1 $W_{-1}(x)$. $W_0(x)$ *principal branch* W . 4.

2. Applications

W P ( ) W . I , W .

Combinatorial applications

. L t_n n . $T(x) =$ $t_n x^n / n!$ fi $T(x) = x + xT(x) + xT(x)^2/2! +$ $=$ $xe^{T(x)}$, $T(x)e^{-T(x)} = x$ $T(x) =$ $W($ $x)$. I [57], R L $t_n = n^{n-1}$, [10,13,67].

$$U(x) = T(x) \quad \tfrac{1}{2}T(x)^2\,; \tag{2:1}$$

$U(x)$ *unrooted* [24,41]. ,

$$V(x) = \tfrac{1}{2} \quad \frac{1}{1 \quad T(x)} \tag{2:2}$$

; $\tfrac{1}{2}T(x) + \tfrac{1}{4}T(x)^2$

Solution of a jet fuel problem

[illegible] E_t [illegible] R [illegible] [3, *pp.* 312-323]. [illegible] c_t [illegible] w_0 [illegible] w_1, [illegible]

$$E_t = \frac{C_L}{c_t C_D} [illegible] \frac{w_0}{w_1} ; \qquad (2{:}6)$$

$$R = \frac{2}{c_t C_D} \frac{2C_L}{S}^{1=2} w_0^{1=2} \quad w_1^{1=2} ; \qquad (2{:}7)$$

[illegible] E_t [illegible], C_L [illegible] C_D [illegible], w_0 [illegible], w_1 [illegible], R [illegible], S [illegible]

$w = w_1 = w_0$ [illegible] $c = E_t C_D c_t = C_L$ [illegible]

$$A = \frac{p\,\overline{2}E_t}{R} \quad \frac{w_0}{SC_L}^{1=2} : \qquad (2{:}8)$$

[illegible] $c =$ [illegible] w [illegible]

$$2A \frac{1 \quad p\,\overline{w}}{[illegible] \ w} = 1 : \qquad (2{:}9)$$

[illegible] $A < 0$, [illegible] w. I [illegible] W, [illegible]

$$w = \begin{cases} A^{-2} W_0^2 \left(Ae^A\right) ; & A \quad 1, \\ A^{-2} W_{-1}^2 \left(Ae^A\right) ; & 1 \quad A < 0. \end{cases} \qquad (2{:}10)$$

O w [illegible], [illegible] $c =$ [illegible] w.

Solution of a model combustion problem

[illegible]

$$\frac{dy}{dt} = y^2(1 \quad y); \quad y(0) = " > 0 \qquad (2{:}11)$$

[54,59] [illegible] W, [illegible] [54] [illegible] $y(t)$ ([illegible] [54]):

$$\frac{1}{y} + [illegible] \frac{1}{y} \quad 1 = \frac{1}{"} + [illegible] \frac{1}{"} \quad 1 \quad t ; \qquad (2{:}12)$$

[illegible]

$$s = W_k(a)\,;$$

[illegible] W_k. ([illegible] 4 [illegible] W.) [illegible] $(W_k(a)t)$ [illegible] $y = ay(t\;\;1)$, [illegible]

$$y = \sum_{k=-\infty}^{\infty} c_k \quad (W_k(a)t)\,; \qquad (2{:}18)$$

[illegible] c_k (. . [illegible]). O [illegible] $W_k(a)$ [illegible] [8]. [illegible] ffi [illegible]

, fi R.

Similarity solution for the Richards equation

W [7]. B , *moisture tension* Ψ,

$$\frac{d}{d\Psi}\frac{@\Psi}{@t} = \frac{@}{@z}\quad K(\Psi)\frac{@\Psi}{@z}\quad K(\Psi)\Big] \; ; \tag{2:24}$$

, , ff

$$^*A\frac{dA}{dt} = 1 \quad A \tag{2:25}$$

W

$$A(t) = 1 + W\quad(\quad 1 + A(0))\qquad \frac{(A(0)\quad 1)\;^*\quad t}{1\quad A(0)}\;\Big] \; : \tag{2:26}$$

B W . W_0 *capillar rise*, W_{-1} *infiltration.*

Volterra equations for population growth

I [22, 102 109], fi

$$\frac{dx}{dt} = ax(1\quad y); \quad \frac{dy}{dt} = \quad cy(1\quad x)\; ; \tag{2:27}$$

W ((11) (12) 104 [22]). , $a = 2$ $c = 1$, B1 DE E ff [40]. . y^+ y^-,

$$\begin{aligned} y^+ &= \quad W_{-1}\quad Cx^{-c=a}e^{cx=a}\quad ; \\ y^- &= \quad W_0\quad Cx^{-c=a}e^{cx=a}\quad ; \end{aligned} \tag{2:28}$$

C .

fi t, fi :

$$t = \int_{x_0}^{x} \frac{d}{a\;(1\quad y(\;))} = \int_{y_0}^{y} \frac{d}{c\;(1\quad x(\;))}\; : \tag{2:29}$$

, - .

$c_0; c_1; ...$

, . I ,
.
I L [12].
W fi .

3. Calculus

W 0. L
(. [12]), $W_0(z)$:

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(\ \ n)^{n-1}}{n!} z^n : \qquad (3{:}1)$$

I we^w.
B W_0 $z = \ \ 1{=}e$ (
$w \mapsto we^w$ 0 $w = \ \ 1$),
(3.1) $1{=}e$. $1{=}e$
.
, E (1.2)

$$M_E(\ ;\ )jvj < \frac{1}{e} ;$$

$M_E(a;b) = \sqrt{(a^2+b^2){=}2}$ E a b. ,
, . F
v, =
. ,
W (1.2)
= = 1 W .
D ff fi $x = W(x)e^{W(x)}$ W W',
W:

$$\begin{aligned} W'(x) &= \frac{1}{(1+W(x)) \quad (W(x))} \\ &= \frac{W(x)}{x(1+W(x))} ; \qquad x \neq 0. \end{aligned} \qquad (3{:}2)$$

H , ,
. , M 3
$W(x){=}(x(1+W(x)))$ ff W,
$W_0'(0) = 1$. [20]

(-) n W

$$\frac{d^n W(x)}{dx^n} = \frac{(\quad nW(x))p_n(W(x))}{(1+W(x))^{2n-1}} \qquad n \quad 1\,; \tag{3.3}$$

$p_n(w)$

$$p_{n+1}(w) = \quad (nw + 3n \quad 1)p_n(w) + (1+w)p_n'(w)\,; \qquad n \quad 1: \tag{3.4}$$

$p_1(w) = 1.$ $p_n(0)$ $(\quad n)^{n-1} = n!$ n 1. A p_n ,

$$\frac{d^n W(e^x)}{dx^n} = \frac{q_n\left(W(e^x)\right)}{\left(1+W(e^x)\right)^{2n-1}} \qquad n \quad 1\,; \tag{3.5}$$

q_n,

$$q_n(w) = \sum_{k=0}^{n-1} \left\langle {n \quad 1 \atop k} \right\rangle (\quad 1)^k w^{k+1}\,; \tag{3.6}$$

ﬃ E [35].

$$q_{n+1}(w) = \quad (2n \quad 1)wq_n(w) + (w+w^2)q_n'(w) \tag{3.7}$$

$q_1(w) = w$.

$We^W = z$, ﬁ

$$W(z) = z \quad W(z)\,; \tag{3.8}$$

$z > 0$. [43]

W .

I [58], K. B. N :

$$x = pe^p\,; \tag{3.9}$$

$$\frac{dy}{dx} = p: \tag{3.10}$$

D ﬀ (3.9) y

$$\frac{dx}{dy} = \frac{dp}{dy}e^p + pe^p\frac{dp}{dy}\,; \tag{3.11}$$

(3.10)

$$\frac{dy}{dp} = p(p+1)e^{p} \; ; \quad (3{:}12)$$

$$y = (p^2 \quad p + 1)e^{p} + C \; : \quad (3{:}13)$$

y $W(x)$, $W(x)$. I ,

$$W(x)\,dx = (W^2(x) \quad W(x) + 1)e^{W(x)} +$$

Figure 3. [illegible] $p =$ [illegible] z. [illegible] $z = e^p$ [illegible] F[illegible] 2. [illegible] C [illegible] D [illegible]

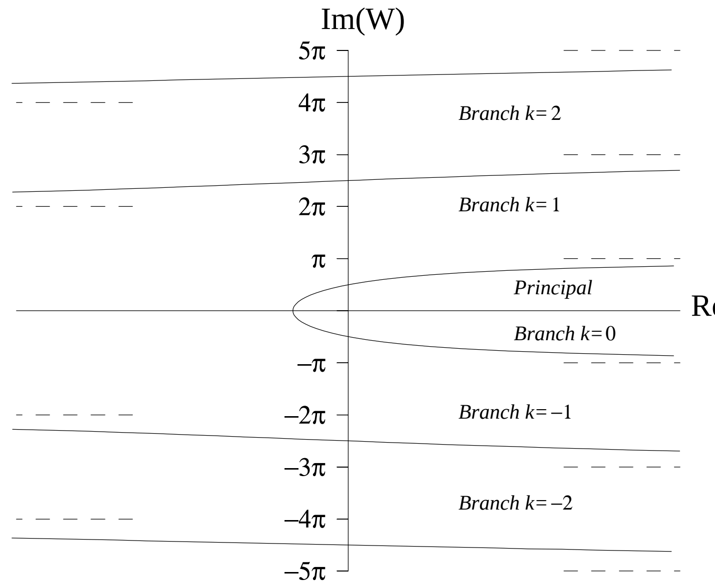
Im(W)
5π
4π
3π
2π
π
−π
−2π
−3π
−4π
−5π
Re
Branch k = 2
Branch k = 1
Principal
Branch k = 0
Branch k = −1
Branch k = −2

Figure 4. [illegible] $W(z)$. [illegible] 0. [illegible]

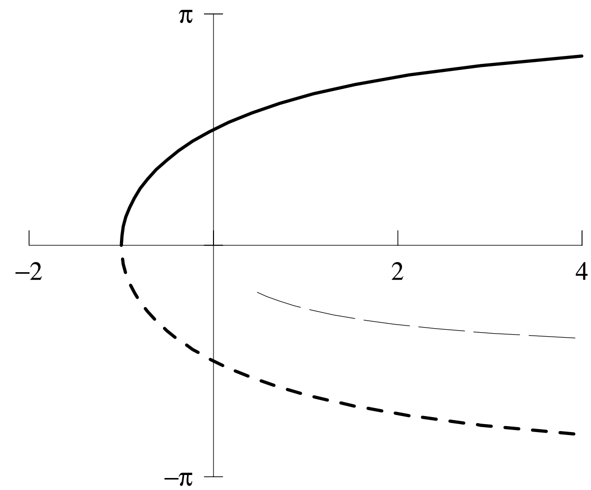

Figure 5. ... , $W_0(z)$. ... A, B, C, ... D ... Figure 6. ... Figure 6.

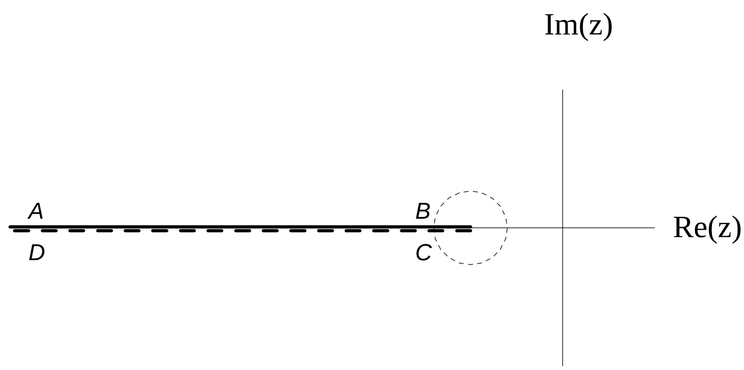

Figure 6. ... $W_0(z)$. ... $z = we^w$ ... Figure 5, ... C ... B ... Figure 5, ... B. ... $\{z : -\infty < z \le -1/e\}$.

... , $W_0(z)$ ... (CCC) ... $z = -1/e$. ... $z = -1/e + \varepsilon e^{it}$ ... $-\pi < t \le \pi$, ... 0. ... Figure 5 ... Figure 6.

B... , W_{-1} ... W_1 ... $\{z : -\infty < z \le -1/e\}$ ... $\{z : -1/e < z \le 0\}$. ... CCC ... $z = 0$. ... $W_{-1}(z)$ ... $z \in [-1/e, 0)$. ... $W_0(z)$ ... $W_{-1}(z)$ ... W ... Figure 7, Figure 8, ... Figure 9. A ... W ... $\{z : -\infty < z \le 0\}$, ... , ...

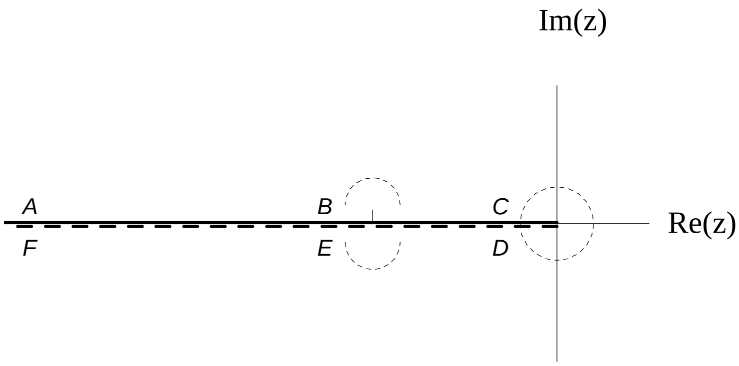

Figure 7. … $W_k(z)$, $k \neq 0$. … $W_1(z)$ … $W_{-1}(z)$, … $z = -1/e$ … $z = we^w$ … Figures 8 and 9. … D … C, … C, … D … C … Figures 8 and 9.

Figure 8. … $W_{-1}(z)$. … A, … $W = -1$ … $z = (-\infty, -1/e]$. … $W = -1$, … C … $z = [-1/e, 0)$. … $W_{-1}(z)$ … CD … .

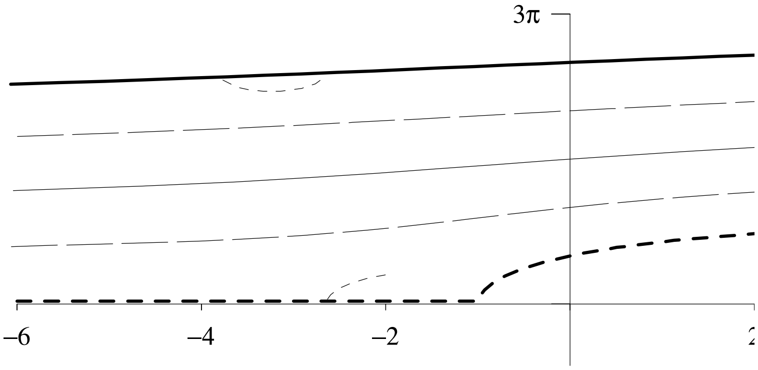

Figure 9. D $W_1(z)$. A C z . N $W_1(z)$, . , .

(4.22), $\{z : -1 < z \le -1/e\}$ $\{z : 0 \le z < \infty\}$.

Asymptotic expansions

W 0 ( ) fi , $w \mapsto we^w$. , $z = 0$.)

$$w = W(z) = z + u\,; \tag{4.6}$$

z 0 u z, fi W, $we^w = z$,

$$(z + u) z e^u = z\,; \tag{4.7}$$

$$(1 + \frac{u}{\;z})e^u = \frac{1}{\;z}\,: \tag{4.8}$$

$|u| \ll |\;z|$ $e^u \sim 1/\;z$,

$$u \sim \frac{1}{\;z}\,; \tag{4.9}$$

(4.9) . H , u fi (4.9), . ,

(4.9) :

$$u \quad \frac{1}{\quad z} : \tag{4.10}$$

O ... z ... $\frac{1}{\mathrm{Log}\, z}$ ... z. ... z ... z ... 1. K ... , ... B ... [11, . 27 28] ... W ... 0 ... fi . (B ... $z \to \infty$, ... $z \to 0$ ... . B ... L ... I ... .) I ... , ... , ...

C ... (4.6) ... (4.10), ...

$$w = \quad z \quad\quad z + v : \tag{4.11}$$

O ... (4.11) ... $we^w = z$ ...

$$\frac{(\quad z \quad\quad z + v)e^v z}{\quad z} = z : \tag{4.12}$$

F ... $1 = \quad z$ ... $z = \quad z$, ... B ... . , ... $z \neq 0$ ... $z \neq 0$,

$$e^{-v} \quad 1 \quad v + \quad = 0 : \tag{4.13}$$

... (2.4.6) B ... ,

Remark

(4:

... fi ... (4.18), ... $z + 2 k$ ... k,

$$W_k(z) = \ln z + 2\pi i k - \ln(\ln z + 2\pi i k) + \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} c_{km} \ln^m(\ln z + 2\pi i k)(\ln z + 2\pi i k)^{-k-m} \quad (4{:}20)$$

... k; ... $z \to 1$, ... A ... $W_{-1}(x)$ ... $x < 0$. ... $(-x)$ ... (z) ... $((-x))$ ... (z). A ... (4.13) ... v, ... A ... ffi ... x ... W:

Series expansions about the branch point

... $p = +\sqrt{2(ez+1)}$ ... $We^W = z$, ... $1 + W$, ...

$$\frac{p^2}{2} - 1 = We^{1+W} = -1 + \sum_{k\geq 1}\left(\frac{1}{(k-1)!} - \frac{1}{k!}\right)(1+W)^k ; \quad (4{:}21)$$

...

$$W(z) = \sum_{\ell=0}^{\infty} \mu_\ell p^\ell = -1 + p - \frac{1}{3}p^2 \ldots$$

F W_{-1} $p = \sqrt{2(ez+1)}$, I (z) 0. F I (z) < 0, $W_1(z)$. 1=e

W_{-1} W_0 [45] K ffi c_n

$$= \;+\tfrac{2}{3}\;{}^{2}+\tfrac{4}{9}\;{}^{3}+\tfrac{44}{135}\;{}^{4}+\;=\sum_{n\geq 1} c_n\;{}^{n}\,; \tag{4:26}$$

$$(1+\;)e^{-}\;=(1\;\;)e\;;\qquad =\;+O(\;^{2})\,: \tag{4:27}$$

[41, 323]. H c_n n 15

$$\frac{1}{n}\quad c_n\quad \frac{1}{n}+\frac{1\quad 1{=}n}{n(1+1{=}2+\quad +1{=}n)}\,: \tag{4:28}$$

I M 3, W . P M 2, 7! e [1; 1) , . I M 2, (1 ; 1=e] (0; 1), fi - fi W M 3.

5. Numerical Analysis

I [27], E $x = a^x$ a , fi $W_k(x)$ x L [51,52]. [36] H fi $W_k(x)$ x , . , [70], [71] .

W : fi , arbitrar , M . fi W, ffi .

(. [16, . 14]) $C = xf'$

ff z.

W $z = -1/e$ W_0, W_{-1} W_1), $W(z)$ 1. I , $z = 0$; , *relative*

. $W($

$jW'(z)j =$. O $W_k(z)$
$W_{-k}(z)$, $\overline{W_k(z)} = W_{-k}(\overline{z})$ (-
4). F ; $W_k(z)$ k

[illegible]

. I ... M ... $W_0(x)$, $1/e$ x, ... $W_{-1}(x)$, $1/e$ $x < 0$, ... A ... W ... F ... W ... , H ...

$$w_{j+1} = w_j - \frac{w_j e^{w_j} - z}{e^{w_j}(w_j+1) - \frac{(w_j+2)(w_j e^{w_j} - z)}{2w_j+2}} : \qquad (5:9)$$

I ... M ... , ... F ... z, ... ffi ... (4.18). ... $W_0(z)$, $W_{-1}(z)$ ... $W_1(z)$ ... z ... $1/e$, ... fi ... (4.22) ... W_{-1} ... W_1). ... $W_0(z)$ ... 0, ... P ... (M ... (3,2)-P ...). F ... , ... $W_{-1}(z)$ ... z ... 0 ... $1/e$ ... W_{-1} ... , ... z ... $1/e$ ... . A ... W_1. F ... M ... , ...

6. Concluding Remarks

... L ... W ... , ... W ... ; ... , ... : A ... W ... (... , ... $T(x)$). N ... . L ... W ... fi ... , ... ff ... , ... [21], ...

Acknowledgements.

A ... , ... ; ... . A ... H ... N ... - ff ... .

B ... ffi ... *W* ... C ... J. B ... (3.11) ... L ... [4]. ... M ... K ... H ... H ... -O. ... G.-C. ... D. L ... P ... D. E. G ... (... L ...) ... E ... [28]; ... M ... D ... F ... N ... L ... *éloge*

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[359]

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F , *éloge* L ’ (1946 1948) fi L (P ’). L A L , .