THE SLOW MOTION OF A CYLINDER NEXT TO A PLANE WALL

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SUMMARY

The two-dimensional flow around a cylinder that is near a plane wall is calculated assuming that the Reynolds number for the flow is small. For a cylinder translating parallel to the wall, the torque on the cylinder is zero; similarly, for a rotating cylinder the force is zero. These results, which are surprising when compared with corresponding ones for a sphere, are proved and then examined further using lubrication theory. We also consider motion perpendicular to the wall, allowing us to discuss the behaviour of a cylinder falling down an inclined plane. In addition streamline patterns are described.

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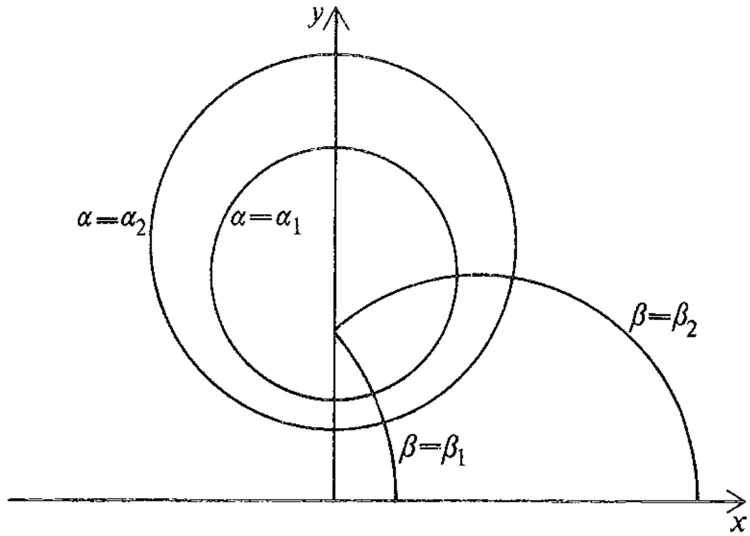

FIG. 1. Bipolar coordinates (α, β). $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$.

are bounded externally or that decay sufficiently rapidly at infinity is

$$h\psi = \mathrm{Re}\left\{\sum_{n=0}^{\infty} \chi_n(\alpha)e^{-in\beta}\right\}, \tag{2}$$

where

$$\chi_0 = A_0 \cosh \alpha + B_0\alpha \cosh \alpha + C_0 \sinh \alpha + D_0\alpha \sinh \alpha,$$
$$\chi_1 = A_1 \cosh 2\alpha + B_1 + C_1 \sinh 2\alpha + D_1\alpha,$$
$$\chi_n = A_n \cosh (n+1)\alpha + B_n \cosh (n-1)\alpha + C_n \sinh (n+1)\alpha + D_n \sinh (n-1)\alpha.$$

The constants are complex (except for $n=0$), and not all independent, because to obtain a single-valued pressure field, we must set $\mathrm{Re}\{D_1\} = -B_0$; the pressure field is then

$$p = \frac{2\mu}{a} \mathrm{Im}\left[e^{-i\beta}\{D_0 \sinh \alpha + (B_0 + D_1) \cosh \alpha\} + \right.$$

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The Cartesian components of the force on the surface $\alpha=\alpha_1$ are

$$F_x\mathbf{i}+F_y\mathbf{j}=\int \boldsymbol{\sigma}\cdot\mathbf{n}\,ds=4\pi\mu D_0\mathbf{i}-4\pi\mu\,\mathrm{Im}\,\{D_1\}\mathbf{j}. \qquad (3)$$

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wall. If the radius of the cylinder is r and the distance of the cylinder axis from the wall is d, then the cylinder is described by $\alpha=\alpha_1$, where

$$d=a\coth\alpha_1,\qquad r=a\,\mathrm{cosech}\,\alpha_1\quad\text{and}\quad a^2=d^2-r^2.$$

2. Particular solutions

2.1 Cylinder rotating next to a plane wall

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non-zero coefficients are

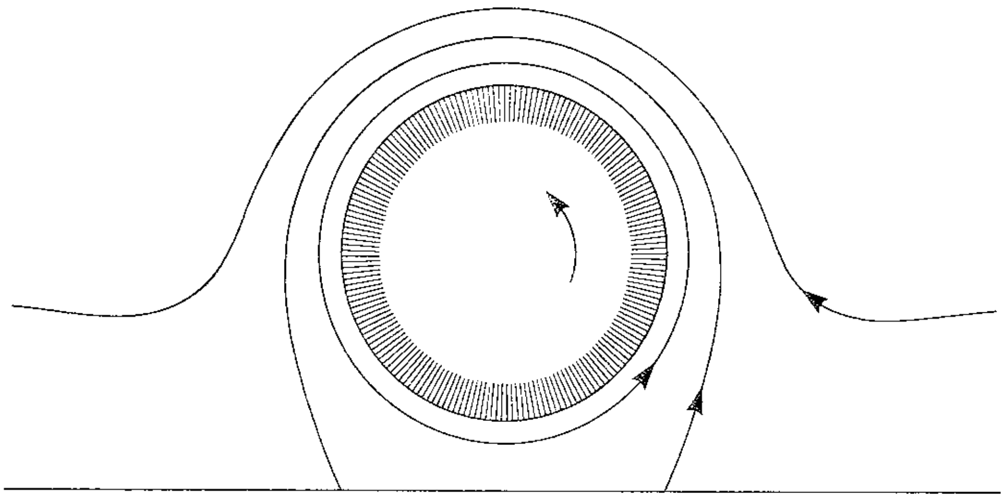

FIG. 2. The streamline pattern for low-Reynolds-number flow around a cylinder rotating next to a plane wall, plotted using a computer graphics package.

flowing towards this point, forcing the fluid coming from infinity to turn around and flow over the cylinder. The possible ways in which fluid can turn [redacted]

possible if two streamlines, rather than one, met the boundary at the point in question (cf. (**8**), Fig. 2b). A check of the derivatives of ψ at this point

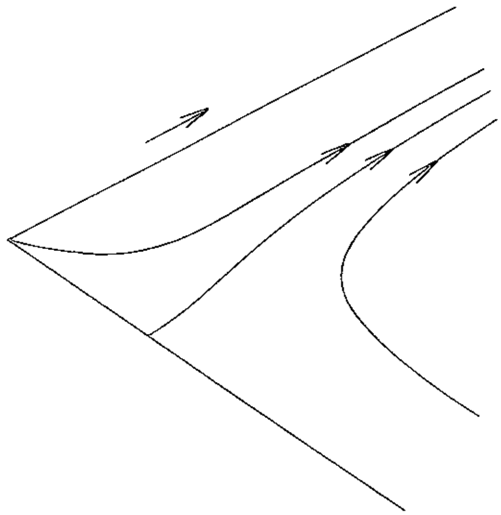

FIG. 3. Flow in a corner when one wall slides parallel to itself and there is a line source of fluid at the vertex.

3.2 The other flows

The streamlines for the other two flows do not vary qualitatively from the streamlines for the special cases of line forces, and these have been given in (**7**). We do note, though, that for the case of motion parallel to the wall, it has been shown in (**4**) that, in the frame of reference in which the cylinder is stationary, a dividing streamline meets the cylinder at $x = \pm ax/d$, $y = a^2/d$

while the boundary conditions $\mathbf{u}.\mathbf{n}=0$ and $\mathbf{u}.\mathbf{t}=\omega r$ become

$$u_0=1 \quad \text{and} \quad v_0=X \quad \text{on} \quad Y=H.$$

The solutions for u_0 and P that make $P\to 0$ as $X\to\infty$ are

$$P=-2X/H^2 \quad \text{and} \quad u_0=\tfrac{1}{2}P'Y(Y-H)+Y/H.$$

Thus the rotation sets up a pressure field which adds a Poiseuille flow to the more obvious Couette flow. The volume flux through the gap is

$$Q=\omega r^2\epsilon\int_0^H u_0\,dY=\tfrac{2}{3}\omega r^2\epsilon,$$

and the torque and force acting on the cylinder are

$$T=-\mu\omega r^2\epsilon^{-\frac{1}{2}}\int_{-\infty}^{\infty}\left[\frac{\partial u_0}{\partial Y}\right]_{Y=H}dX+O(1)=-4\pi\mu\omega r^2(2\epsilon)^{-\frac{1}{2}}+O(1)$$

and

$$F_x=-\mu\omega r\epsilon^{-\frac{1}{2}}\int_{-\infty}^{\infty}\left\{\left[\frac{\partial u_0}{\partial Y}\right]_{Y=H}+XP\right\}dX=0.$$

We see that F_x is zero because the pressure field provides a force on the cylinder which exactly balances the skin friction $\partial u_0/\partial Y$. Also, $\partial u_0/\partial Y$ on the plane is zero at $X=\pm\sqrt{2}$, indicating a separation point. All these results agree with the appropriate limits taken from the general solution.

4.2 Motion tangential to wall

We scale the velocities and pressure according to

$$u=Uu_0+O(\epsilon), \qquad v=U\epsilon^{\frac{1}{2}}v_0+O(\epsilon^{\frac{3}{2}}) \quad \text{and} \quad p=\mu Ur^{-1}\epsilon^{-\frac{3}{2}}P+O(\epsilon^{-\frac{1}{2}}).$$

The boundary conditions become $u_0=1$ and $v_0=0$, but the equations are

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$$P=-6/H^2 \quad \text{and} \quad u_0=\tfrac{1}{2}P'Y(Y-H).$$

The force is

$$F_y=\mu V\epsilon^{-\frac{3}{2}}\int_{-\infty}^{\infty} P\,dX=-12\pi\mu V(2\epsilon)^{-\frac{3}{2}}+O(\epsilon^{-\frac{1}{2}}),$$

so that actually the next term in the approximation is also singular.

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Thus the sphere rotates four times slower than one's everyday ideas of rolling would predict. If the centre of the sphere is at $(x_c, r+r\varepsilon, 0)$, then

$$\varepsilon = \epsilon_0 \exp(-kt) + O(te^{-2kt}) \quad \text{and} \quad x_c = 2r \tan\theta \log(kt) + O(1),$$

where $k = mg\cos\theta/6\pi\mu r^2$ and ϵ_0 is a notional initial value. The fact that x_c is logarithmic in time for both sphere and cylinder is a surprising coincidence.

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