Comparison of homotopy analysis method and homotopy perturbation method through an evolution equation

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Abstract

In this paper, the homotopy analysis method (HAM) proposed by Liao in 1992 and the homotopy perturbation method (HPM) proposed by He in 1998 are compared through an evolution equation used as the second example in a recent paper by Ganji *et al* (2007). It is found that the HPM is a special case of the HAM when ~ $= -1$. However, the HPM solution is divergent for all x and t except $t = 0$. It is also found that the solution given by the variational iteration method (VIM) is divergent too. On the other hand, using the HAM, one obtains convergent series solutions which agree well with the exact solution. This example illustrates that it is very important to investigate the convergence of approximation series. Otherwise, one might get useless results.

Key words: Homotopy analysis method (HAM), analytical solution, convergence, symbolic computation
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1 Introduction

Many physics and engineering problems can be modelled by di®erential equations. However, it is di±cult to obtain closed-form solutions for them, especially for nonlinear ones. In most cases, only approximate solutions (either is based on the existence of small/large parameters, the so-called perturbation quantities [1]. However, many nonlinear problems do not contain such kind of perturbation quantities. In general, the perturbation method is valid only for weakly nonlinear problems. For example, considering the following heat transfer problem [2] governed by the nonlinear ordinary di®erential equation

$$(1 + ^2 u)u^0 + u = 0; \quad u(0) = \text{tial}; \quad \text{u}$$

for weakly nonlinear problems in general. In view of the work by Abbasbandy [2], we see that the HAM allows us to extend a series approximation beyond its initial radius of convergence.

Fig. 1. Comparison of the exact and approximate solutions of (1). Solid line: exact solution u^0

equation [8]:

$$u_t + u_x = 2u_{xxt}, \qquad x \in \mathbb{R}, t > 0, \tag{10}$$

$$u(x, 0) = e^{-x}. \tag{11}$$

Its exact solution reads

$$u_{exact}(x, t) = e^{-x-t}. \tag{12}$$

By means of the homotopy perturbation method, Ganji *et al* [8] rewrote the original equation in the following form

$$(1-p)\,\frac{\partial \phi(x,t;p)}{\partial t} + p\left[\frac{\partial \phi(x,t;p)}{\partial t} + \frac{\partial \phi(x,t;p)}{\partial x} - 2\,\frac{\partial^3 \phi(x,t;p)}{\partial x^2 \partial t}\right] = 0, \tag{13}$$

subject to the initial condition

$$\phi(x, 0; p) = e^{-x}, \tag{14}$$

where $p \in [0, 1]$ is an embedding parameter. Then, regarding p as a small parameter, Ganji *et al* [8] expanded $\phi(x, t; p)$ in a power series

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)\, p^m, \tag{15}$$

which gives the solution by setting $p = 1$. Substituting (15) into the original equation and initial condition, then equating the coefficients of the like powers of p, one can get governing equations and the initial conditions for $u_m(x, t)$. In this way, Ganji *et al* [8] obtained the mth-order approximation

$$u(x,t) \approx u_0(x,t) + \sum_{k=1}^{m} u_k(x,t), \tag{16}$$

and the 5th-order approximation reads

$$u_{HPM}(x,t) \approx \frac{e^{-x}}{720}(t^6 + 66t^5 + 1470t^4 + 13320t^3 + 46440t^2 + 45360t + 720). \tag{17}$$

However, for *any* given $x \geq 0$, the above approximation enlarges monotonously to the positive infinity as the time t increases, as shown in Fig. 2. Unfortunately, the exact solution monotonously decreases to zero! Let

$$\delta(t) = \left|\frac{u_{exact} - u_{HPM}}{u_{exact}}\right| \tag{18}$$

denote the relative error of the HPM approximation (17). As shown in Fig. 2, the relative error $\delta(t)$ monotonously increases very quickly:

$$\delta(0) = 0, \quad \delta(0.1) = 7.8, \quad \delta(1) = 404.4, \quad \delta(10) = 1.25 \times 10^9.$$

Fig. 2. Approximations of (10) given by the homotopy perturbation method. Dashed-line: exact solution (12); Solid line: the 5th-order HPM approximation (17); Dash-dotted line: the relative error $\pm(t)$ de¯ned by (18); Hollow symbols: the 40th-order HAM approximation (42) when ~ = 1.

2 HAM solution versus HPM solution

In order to solve (10) and (11) by means of the HAM, we first construct the zeroth-order deformation equation

$$(1-p)\,\frac{\partial \phi(x;t;p)}{\partial t} = p\,\hbar \left[\frac{\partial \phi(x;t;p)}{\partial t} + \frac{\partial \phi(x;t;p)}{\partial x} - 2\,\frac{\partial^3 \phi(x;t;p)}{\partial x^2 \partial t} \right]; \qquad (19)$$

subject to the initial condition

$$\phi(x;0;p) = e^{ix}; \qquad (20)$$

where $p \in [0;1]$ is an embedding parameter and $\hbar \neq 0$;p)]F31/F3lE5.870TD[())]TJET0.4w310 and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

The solution of the mth-order deformation equation (24) for $m \geq 1$ reads

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar \int_0^t R_m(u_{m-1}(x,t))\, d\tau + c_1, \tag{27}$$

where the constant of integration c_1 is determined by the initial condition (25).

Using symbolic computation systems such as Maple or Mathematica, we recursively obtain

$$u_0(x,t) = e^{-x}, \tag{28}$$

$$u_1(x,t) = -\hbar e^{-x} t, \tag{29}$$

$$u_2(x,t) = \frac{\hbar e^{-x} t}{2} (\hbar t + 2\hbar - 2), \tag{30}$$

$$u_3(x,t) = -\frac{\hbar e^{-x} t}{6} \left(\hbar^2 t^2 - 6\hbar t + 6\hbar^2 t + 6\hbar^2 - 12\hbar + 6 \right), \tag{31}$$

$$\begin{aligned} u_4(x,t) = \frac{\hbar e^{-x} t}{24} \big(& \hbar^3 t^3 - 12\hbar^2 t^2 + 12\hbar^3 t^2 + 36\hbar t \\ & -72\hbar^2 t + 36\hbar^3 t - 24 + 72\hbar - 72\hbar^2 + 24\hbar^3 \big), \end{aligned} \tag{32}$$

$$\begin{aligned} u_5(x,t) = -\frac{\hbar e^{-x} t}{120} \big(& \hbar^4 t^4 - 20\hbar^3 t^3 + 20\hbar^4 t^3 + 120\hbar^2 t^2 - 240\hbar^3 t^2 \\ & +120\hbar^4 t^2 + 720\hbar^2 t - 240\hbar t + 240\hbar^4 t - 720\hbar^3 t - 480\hbar^3 \\ & -480\hbar + 120 + 720\hbar^2 + 120\hbar^4 \big), \end{aligned} \tag{33}$$

$$\begin{aligned} u_6(x,t) = \frac{\hbar e^{-x} t}{720} \big(& \hbar^5 t^5 - 30\hbar^4 t^4 + 30\hbar^5 t^4 + 300\hbar^5 t^3 + 300\hbar^3 t^3 \\ & -600\hbar^4 t^3 - 3600\hbar^4 t^2 + 1200\hbar^5 t^2 - 1200\hbar^2 t^2 + 3600\hbar^3 t^2 \\ & -7200\hbar^2 t + 1800\hbar t - 7200\hbar^4 t + 10800\hbar^3 t + 1800\hbar^5 t \\ & +7200\hbar^3 + 3600\hbar - 720 - 7200\hbar^2 + 720\hbar^5 - 3600\hbar^4 \big). \end{aligned} \tag{34}$$

When $\hbar = -1$, it is easily seen that the equations (30) up to (34) above are exactly the equations (3.17b) up to (3.17f) in [8], and the combination of the equations (28) and (29) is exactly the equation (3.17a) in [8] (Ganji *et al* made mistakes in the first two lines of (3.16) in [8], which makes the difference). Furthermore, when $\hbar = -1$, the 6th-order approximation

$$u(x,t) \approx \frac{e^{-x}}{720}(t^6 + 66t^5 + 1470t^4 + 13320t^3 + 46440t^2 + 45360t + 720) \tag{35}$$

is exactly the *same* as the HPM solution (17). Therefore, the HPM solution is indeed a special case of the HAM solution when $\hbar = -1$. This fact has been pointed out by many researchers, such as Abbasbandy [2], Liao *et al*

[12], Bataineh *et al* [14], Van Gorder *et al* [15], Hayat and Sajid [16][17], and Song *et al* [18].

Unfortunately, ~ = -1 is *not* a proper value for the current problem, because

Fig. 4. The ~-curve for 30th-order HAM approximation of $u(0;1)$.

$$\sum_{i=0}^{1} u_i$$

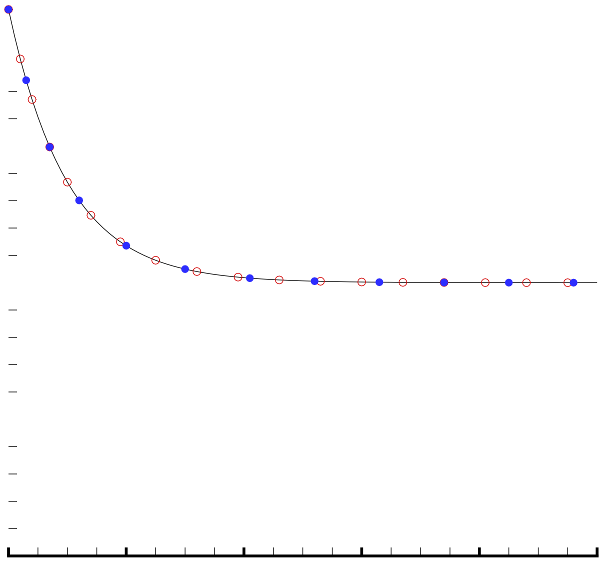

Fig. 5. Comparison of the HAM approximation with the exact solution. Solid line: the exact solution; Hollow symbols: the HAM result when $\hbar = 1/2$; Filled symbols: the HAM result when $\hbar = 3/4$.

3 Conclusion

In this paper, we compare the homotopy analysis method (HAM) proposed by Liao [9] in 1992 and the homotopy perturbation method (HPM) proposed by He [6] in 1998 through a linear partial differential equation. It is found that the HPM is indeed a special case of the HAM when $\hbar = -1$. However, the HPM result is divergent for *all* x and t except $t = 0$ (which is given as the initial condition), i.e. the convergence radius of the HPM solution is zero. Note that, using the variational iteration method (VIM), one obtains exactly the *same* result as the HPM solution (17). Therefore, the HPM (as well as the VIM) solution does not provide a useful approximation, either in the sense of convergent series, or in the sense of asymptotic series. This example illustrates that it is very important to obtain knowledge of the accuracy of any approximation.

It is true that, like other non-perturbation techniques such as Lyapunov's artificial small parameter method [3] and Adomian's decomposition method [5], the HPM can give approximations even if a problem does not contain any small/large physical parameters. However, the example above indicates that this is *not* the key point for solving nonlinear problems: using the HPM, one might get divergent results even for a *linear* problem.

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