New travelling wave solutions to modi¯ed CH and DP equations

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Abstract

A new procedure for ¯nding exact travelling wave solutions to the modi¯ed Camassa-Holm and Degasperis-Procesi equations is proposed. It turns out that many new solutions are obtained. Furthermore, these solutions are in general forms, and many known solutions to these two equations are only special cases of them.

Key words: Camassa-Holm equation, Degasperis-Procesi equation, travelling wave solution, tanh method, symbolic computation
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1 Introduction

For the function $u(x;t)$, the Camassa-Holm (CH) equation

u_t ¡ *uat ion*

solutions. The name \peakon", which means travelling wave with slope discontinuities, is used to distinguish them from general travelling wave solutions since they have a corner at the peak of height c, where c is the wave speed.

Since the CH and DP equations have rich structures, Wazwaz [11] suggested a modi¯ed form of the Camassa-Holm equation (called mCH)

$$u_t ¡ u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + u u_{xxx} \tag{3}$$

and a modi¯ed form of the Degasperis-Procesi equation (called mDP)

$$u_t ¡ u_{xxt} + 4u^2 u_x = 3u_x u_{xx} + u u_{xxx}: \tag{4}$$

They are obtained by changing the nonlinear convection term uu_x in equations
^{2}u

of the procedure are presented and discussed. In Section 3, new travelling wave solutions to the mCH equation are obtained. In Section 4, new travelling wave solutions to the mDP equation are obtained. Finally, in Section 5, some technical explanations for the procedure are given.

2 The procedure

The proposed procedure is based on the tanh method [20–23]. We select a list of functions instead of just tanh for finding travelling wave solutions. The list of functions we choose is: [rational, exp, csch, sech, tanh, csc, sec, tan, cn, sn], where cn and sn are Jacobi elliptic functions.

The main steps of the procedure are as follows, where *pde* is the mCH equation (3) or the mDP equation (4), and f is one of the functions in the list above.

S1 Substituting $u(x, t) = U(\xi)$, where $\xi = \lambda_1 x + \lambda_2 t$, into *pde* gives an ODE *ode* with dependent variable $U(\xi)$. The reason why we use $\xi = \lambda_1 x + \lambda_2 t$ instead of $\xi = x + \lambda t$ will be explained in Section 5.
S2 Find the balancing number m of *ode* which is the highest exponent in $T = f(\xi)$ obtained by substituting $U(\xi) = T^m$ into *ode* and then balancing the highest degree terms of T. First, we determine the degree of each term in *ode* with respect to T. Since the degree of $d^pU(\xi)/d\xi^p$ with respect to T is $m+p$ and the degree of $U(\xi)^q$ with respect to T is qm, we obtain a list of term degrees for *ode* in the form of $[c_1m + d_1, \ldots, c_km + d_k]$. Then, we find the degree with maximum value of c and the degree with maximum value of d, and then by equating them we obtain the balancing number m.
S3 Substituting $U(\xi) = \sum_{i=-m}^{m} a_iT^i$ into *ode* and eliminating the common denominator gives an equation. It is easily seen that for every function f in the function list above, any order derivative of $f(\xi)$ with respect to ξ is either a polynomial in $f(\xi)$ or of the form $\Psi\sqrt{\Gamma}$, where Ψ and Γ are polynomials in $f(\xi)$. Therefore, the resulting equation is of the form

$$\Phi + \Psi\sqrt{\Gamma} = 0, \tag{5}$$

where Φ, Ψ and Γ are polynomials in $f(\xi)$.
S4 Setting all the coefficients of the different powers of T in Φ and Ψ of (5) to zero gives a system of polynomial equations.
S5 Solving the system of polynomial equations leads to the determination of the parameters $a_0, a_i, a_{-i} (i = 1, \ldots, m), \lambda_1$, and λ_2.
S6 Substituting the solutions obtained into $U(\xi) = \sum_{i=-m}^{i=m} a_iT^i$ gives the travelling wave solutions of f type.

As an explanation of the procedure, let f be the function csc, and *pde* be the mCH equation (3).

$$
\begin{aligned}
3\lambda_1 a_1 a_2^2 - 10\lambda_1^3 a_1 a_2 &= 0;\\
\lambda_1 a_2^3 - 8\lambda_1^3 a_2^2 &= 0;\\
\lambda_1 a_{-2}^2 a_{-1} &= 0;\\
\lambda_1 a_{-2}^3 &= 0.
\end{aligned}
$$

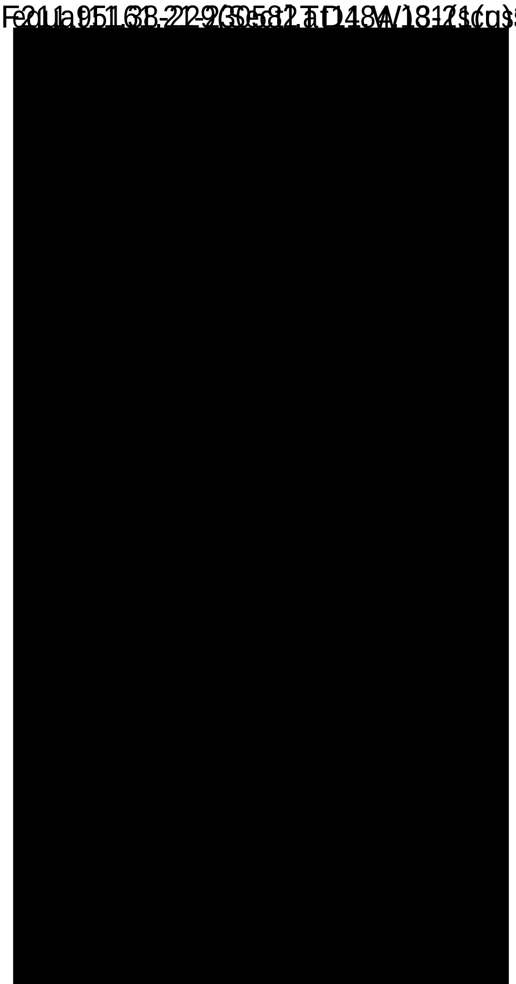
In step S5, solving the system above leads to the following solution:

$$
\begin{aligned}
&a_{-2} = 0; \quad a_{-1} = 0; \quad a_0 = a_0; \quad a_1 = 0;\\
&a_2 = -1 - 2a_0 + \sqrt{1 - 2a_0 - 2a_0^2};\\
&\lambda_1 = \tfrac{1}{4}\sqrt{-2 - 4a_0 + 2\sqrt{1 - 2a_0 - 2a_0^2}};\\
&\lambda_2 = \tfrac{1}{8}\left(\sqrt{-2 - 4a_0 + 2\sqrt{1 - 2a_0 - 2a_0^2}}\right)^3\\
&\qquad + \tfrac{3}{4}\sqrt{-2 - 4a_0 + 2\sqrt{1 - 2a_0 - 2a_0^2}}\, a_0.
\end{aligned} \tag{9}
$$

Finally in step S6, substituting the solution (9) into (7) gives a csc type solution to the mCH equation which is the solution $u_8(x,t)$ in Section 3.

3 Solutions to the mCH equation

The procedure in Section 2 has been implemented by minor modification to the software



$$\alpha = 1 + 2k + \sqrt{1 - 2k - 2k^2},$$
$$\gamma = \frac{1}{4}\sqrt{2\alpha}\,(3k - \alpha).$$

- Three tanh/coth type solutions:

$$u_5(x,t) = k + \frac{1}{18}\alpha \tanh^2\left(\frac{1}{12}\sqrt{\alpha}\,x + \psi t\right), \tag{14}$$
$$u_6(x,t) = k + \frac{1}{18}\alpha \coth^2\left(\frac{1}{12}\sqrt{\alpha}\,x + \psi t\right), \tag{15}$$
$$u_7(x,t) = k + \frac{1}{8}\beta \coth^2\left(\frac{1}{8}\sqrt{\beta}\,x + \phi t\right) + \frac{1}{8}\beta \tanh^2\left(\frac{1}{8}\sqrt{\beta}\,x + \phi t\right), \tag{16}$$

where

$$\alpha = -12 - 24k + 6\sqrt{4 - 2k - 2k^2},$$
$$\beta = -2 - 4k + 2\sqrt{1 - 8k - 8k^2},$$
$$\psi = \frac{1}{108}\sqrt{\alpha}\,(27k + \alpha),$$
$$\phi = \frac{1}{32}\sqrt{\beta}\,(12k + \beta).$$

- One csc type solution and one sec type solution::

$$u_8(x,t) = k + \alpha \csc^2\left(\frac{1}{4}\sqrt{2\alpha}\,x + \gamma t\right), \tag{17}$$
$$u_9(x,t) = k + \alpha \sec^2\left(\frac{1}{4}\sqrt{2\alpha}\,x + \gamma t\right), \tag{18}$$

where

$$\alpha = -1 - 2k + \sqrt{1 - 2k - 2k^2},$$
$$\gamma = \frac{1}{4}\sqrt{2\alpha}\,(3k + \alpha).$$

- Three tan/cot type solutions:

$$u_{10}(x,t) = k + \frac{1}{18}\alpha \cot^2\left(\frac{1}{12}\sqrt{\alpha}\,x + \psi t\right), \tag{19}$$
$$u_{11}(x,t) = k + \frac{1}{18}\alpha \tan^2\left(\frac{1}{12}\sqrt{\alpha}\,x + \psi t\right), \tag{20}$$
$$u_{12}(x,t) = k + \frac{1}{8}\beta \cot^2\left(\frac{1}{8}\sqrt{\beta}\,x + \phi t\right) + \frac{1}{8}\beta \tan^2\left(\frac{1}{8}\sqrt{\beta}\,x + \phi t\right), \tag{21}$$

where

$$\alpha = 12 + 24k + 6\sqrt{4 - 2k - 2k^2};$$
$$\beta = 2 + 4k + 2\sqrt{1 - 8k - 8k^2};$$
$$\psi = \frac{1}{108}\sqrt{\alpha}\,(27k - \alpha);$$
$$\phi = \frac{1}{32}\sqrt{\beta}\,(12k - \beta).$$

• Three cn type solutions:

$$u_{13}(x;t) = \gamma + \frac{1}{6}\sqrt{\beta} + 8k^2\left(1 - \omega^2\right)\mathrm{cn}^{-2}(kx + \psi t;\omega); \qquad (22)$$
$$u_{14}(x;t) = \gamma + \frac{1}{6}\sqrt{\beta} - 8k^2\omega^2\,\mathrm{cn}^2(kx + \psi t;\omega); \qquad (23)$$
$$u_{15}(x;t) = \gamma + \frac{1}{6}\sqrt{\alpha} + 8k^2\left(1 - \omega^2\right)\mathrm{cn}^{-2}(kx + \phi t;\omega) - 8k^2\omega^2\,\mathrm{cn}^2(kx + \phi t;\omega); \qquad (24)$$

where ω cn 7f2,J/22.958.58TD[(1)]TJET0.4w133.28-222.96m139.13-222.96lSBT133490s148(7)8680.148

$$\alpha = 9 - 128k^4\left(1 - 16\omega^2 + 16\omega^4\right);$$
$$\beta = 9 - 128k^4\left(1 - \omega^2 + \omega^4\right);$$
$$\gamma = -\frac{1}{2} - \frac{8}{3}k^2\left(1 - 2\omega^2\right);$$
$$\psi = 8k^3(1 - 2\omega^2) + 3k\left(\gamma + \frac{1}{6}\sqrt{\beta}\right);$$
$$\phi = 8k^3(1 - 2\omega^2) + 3k\left(\gamma + \frac{1}{6}\right.$$
+-(γ)]TJ/F2 11.95 Tf 9.38 0 TD[(+)]TJ 12.8.73 -490s148

$$\alpha = 9 - 128k^4\left(1 + 14\omega^2 + \omega^4\right);$$
$$\beta = 9 - 128k^4\left(1 - \omega^2 + \omega^4\right);$$
$$\gamma = -\frac{1}{2} - \frac{8}{3}k^2\left(1 + \omega^2\right);$$
$$\phi = 8k^3\left(1 + \omega^2\right) + 3k\left(\gamma + \frac{1}{6}\sqrt{\beta}\right);$$
$$\psi = 8k^3\left(1 + \omega^2\right) + 3k\left(\gamma + \frac{1}{6}\sqrt{\alpha}\right);$$

It is noted that the wave speeds of the known travelling wave solutions to the mCH equation in the literature are some specific numbers, while the wave speeds of the solutions from $u_3(x,t)$ to $u_{18}(x,t)$ above are in general forms. In other words, they can be expressed as $c = \frac{F(k)}{G(k)}$, where $F(k)$ and $G(k)$ are some expressions with radicals in k, and k is an arbitrary constant. Consequently, many known travelling wave solutions to the mCH equation are only special cases of them.

For example, if $k = -1$, then $u_6(x,t)$ and $u_7(x,t)$ become

$$u_6(x,t) = -1 + \frac{4}{3}\coth^2\frac{\sqrt{6}}{18}(3x - t); \tag{28}$$
$$u_7(x,t) = -1 + \frac{1}{2}\tanh^2\frac{1}{4}(x - 2t) + \frac{1}{2}\coth^2\frac{1}{4}(x - 2t); \tag{29}$$

which are the solution (1.7) in [13] and the solution (59) in [12] respectively.

If $k = 0$, then $u_3(x,t)$, $u_4(x,t)$ and $u_{12}(x,t)$ become

$$u_3(x,t) = 2\,\mathrm{csch}^2\frac{1}{2}(x - 2t); \tag{30}$$
$$u_4(x,t) = -2\,\mathrm{sech}^2\frac{1}{2}(x - 2t); \tag{31}$$
$$u_{12}(x,t) = \frac{1}{2}\tan^2\frac{1}{4}(x - t) + \frac{1}{2}\cot^2\frac{1}{4}(x - t); \tag{32}$$

which respectively are the solutions (58), (57) and (62) in [12].

If $k = 1$, $u_{10}(x,t)$ and $u_{11}(x,t)$ become

$$u_{10}(x,t) = 1 + 2\cot^2\frac{1}{2}(x - t); \tag{33}$$
$$u_{11}(x,t) = 1 + 2\tan^2\frac{1}{2}(x - t); \tag{34}$$

which respectively are the solutions (61) and (60) in [12].

More real solutions to the mCH equation can be obtained by taking different values for k. Complex solutions can also be obtained by taking suitable values for k, in other words, selecting those values for k such that the values inside the radicals are negative.

4 Solutions to the mDP equation

By the same procedure, we obtain the following travelling wave solutions to the modified DP equation. All the solutions have been verified.

- Two rational type solutions:

$$u_{19}(x;t) = \frac{15}{2x^2}, \tag{35}$$

$$u_{20}(x;t) = \frac{15}{2(x-4t)^2} - 1. \tag{36}$$

- One csch type solution and one sech type solution:

$$u_{21}(x;t) = k + \frac{3}{160}\beta\,\mathrm{csch}^2\left(\frac{1}{20}\sqrt{\beta}\,x + \gamma t\right), \tag{37}$$

$$u_{22}(x;t) = k - \frac{3}{160}\beta\,\mathrm{sech}^2\left(\frac{1}{20}\sqrt{\beta}\,x + \gamma t\right) \tag{38}$$

where

$$\beta = 50 + 100k + 10\sqrt{25 - 60k - 60k^2},$$
$$\gamma = \frac{1}{800}\sqrt{\beta}\,(160k - \beta).$$

- Three tanh/coth type solutions:

$$u_{23}(x;t) = k + \frac{3}{5290}\beta\tanh^2\left(\frac{1}{115}\sqrt{\beta}\,x + \phi t\right), \tag{39}$$

$$u_{24}(x;t) = k + \frac{3}{5290}\beta\coth^2\left(\frac{1}{115}\sqrt{\beta}\,x + \phi t\right), \tag{40}$$

$$u_{25}(x;t) = k + \frac{3}{11560}\alpha\coth^2\left(\frac{1}{170}\sqrt{\alpha}\,x + \psi t\right) + \frac{3}{11560}\alpha\tanh^2\left(\frac{1}{170}\sqrt{\alpha}\,x + \psi t\right), \tag{41}$$

where

$$® = ¡850 \; ¡ \; 1700k + 170\sqrt{25 \; ¡ \; 240k \; ¡ \; 240k^2};$$
$$¯ = ¡1150 \; ¡ \; 2300k + 230\sqrt{25 \; ¡ \; 15k \; ¡ \; 15k^2};$$
$$Á = \frac{4}{304175}$$

$$u_{31}(x;t) = \gamma + \frac{1}{2}\sqrt{\beta} + \frac{15}{2}k^2(1-\omega^2)\,\mathrm{cn}^{-2}(kx+\phi t;\omega)\,; \qquad (47)$$

$$u_{32}(x;t) = \gamma + \frac{1}{2}\sqrt{\beta} - \frac{15}{2}k^2\omega^2\,\mathrm{cn}^{2}(kx+\phi t;\omega)\,; \qquad (48)$$

$$u_{33}(x;t) = \gamma + \frac{1}{2}\sqrt{\alpha} + \frac{15}{2}k^2(1-\omega^2)\,\mathrm{cn}^{-2}(kx+\psi t;\omega) - \frac{15}{2}k^2\omega^2\,\mathrm{cn}^{2}(kx+\psi t;\omega)\,; \qquad (49)$$

where

$$\alpha = 1 - 15k^4\left(1 - 16\omega^2 + 16\omega^4\right);$$
$$\beta = 1 - 15k^4\left(1 - \omega^2 + \omega^4\right);$$
$$\gamma = -\frac{1}{2} - \frac{5}{2}k^2\left(1 - 2\omega^2\right);$$
$$\phi = 10k^3(1-2\omega^2) + 4k\left(\gamma + \frac{1}{2}\sqrt{\beta}\right);$$
$$\psi = 10k^3(1-2\omega^2) + 4k\left(\gamma + \frac{1}{2}\sqrt{\alpha}\right).$$

- Three sn type solutions:

$$u_{34}(x;t) = \gamma + \frac{1}{2}\sqrt{\alpha} + \frac{15}{2}k^2\,\mathrm{sn}^{-2}(kx+\phi t;\omega)\,; \qquad (50)$$

$$u_{35}(x;t) = \gamma + \frac{1}{2}\sqrt{\alpha} + \frac{15}{2}k^2\omega^2\,\mathrm{sn}^{2}(kx+\phi t;\omega)\,; \qquad (51)$$

$$u_{36}(x;t) = \gamma + \frac{1}{2}$$

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$F(k)$ and $G(k)$ are some expressions with radicals in k, and k is an arbitrary constant. Consequently, many known travelling wave solutions to the mDP equation are only special cases of them.

For example, if $k = -\frac{11}{16}$, then $u_{24}(x, t)$ becomes

$$u_{24} = -\frac{11}{16} + \frac{15}{16}\coth^2 \frac{\sqrt{2}}{16}(4x - t), \tag{53}$$

which is the solution (1.11) in [13].

If $k = -\frac{15}{16}$, then $u_{25}(x, t)$ becomes

$$u_{25}(x, t) = -\frac{15}{16} + \frac{15}{32}\tanh^2 \frac{1}{8}(2x - 5t) + \frac{15}{32}\coth^2 \frac{1}{8}(2x - 5t), \tag{54}$$

which is the solution (28) in [12].

If $k = 0$, then u_{21} and u_{22} become

$$u_{21} = \frac{15}{8}\operatorname{csch}^2 \frac{1}{4}(2x - 5t), \tag{55}$$

$$u_{22} = -\frac{15}{8}\operatorname{sech}^2 \frac{1}{4}(2x - 5t), \tag{56}$$

which are the solutions (27) and (26) in [12] respectively.

As in the case of mCH equation, many other real and complex solutions to the mDP equation can be obtained by taking suitable values for k.

5 Conclusions

In this paper, a new procedure has been proposed for finding the exact travelling wave solutions to the modified Camassa-Holm and Degasperis-Procesi equations. Many new solutions have been obtained. Most importantly, these solutions are in general forms and many known solutions to these two equations in the literature are only special cases of them. There are two important technical points in this paper.

First, we use the transformation $\xi = \lambda_1 x + \lambda_2 t$ instead of $\xi = x + \lambda t$ in step S1. The main reason is that, although $\xi = x + \lambda t$ has fewer parameters than $\xi = \lambda_1 x + \lambda_2 t$, the corresponding wave speed has the form $\lambda = \frac{F(k)}{G(k)}$ which is more difficult to solve than $\lambda_1 = G(k)$ and $\lambda_2 = F(k)$. For the example in

Section 2, if the transformation $\xi = x + \lambda t$ is used, then only trivial solutions are obtained in step S5.

Second, in order to get general forms of the travelling wave solutions, we do not integrate the resulting ODE and set the constant of integration to zero in step S1. Otherwise, only special solutions can be obtained. For example, let f be the function csc and *pde* the mCH equation (3) as in Section 2. If we integrate the resulting ODE (6) and set the constant of integration to zero, then, instead of obtaining the general form $u_8(x,t)$ in Section 3, we only obtain the following special solution with wave speed $c = 1$:

$$u(x,t) = -1 + 2\csc^2 \frac{1}{2}(x - t). \tag{57}$$

The reason is as follows. The constant of integration is an arbitrary constant which corresponds to the general form of solution. Therefore, when it is set to zero, only a special solution is obtained.

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