In this paper, we use the computer-algebra system Maple to investigate the properties of series expansions for $W$. We focus on a number of asymptotic expansions for large $z$; these are also valid for non-principal branches around $z = 0$. One practical application of the series is to provide initial estimates for the numerical evaluation of $W$; these estimates can then be refined using iterative schemes to provide the arbitrary precision computations used in computer algebra systems. The series also have intrinsic interest. For example, the definition above of the branches $W_k$ is based on partitioning the plane using the asymptotic series. Another interest is the fact that the asymptotic series are also convergent, and the nature of the convergence is one particular interest of this paper. In this paper, we shall mostly be concerned with the principal branch $k = 0$, which is the only branch that is finite at the origin. We shall abbreviate $W_0$ to $W$ for the rest of the article.

The first asymptotic series is that found by de Bruijn [4] and Comtet [5] as

$$W(z) = \ln z - \ln\ln z + u\ ; \tag{3}$$

where $u$ has the series development

$$u = \sum_{n=1}^{1} \sum_{m=1}^{n} \quad =$$

Wψψψψψψd/(95]TJF1029.963Tf9.960Td[(7)]TJF29L963Tf-1.38-21.16(780)

(951TJF1029.963Tf9.960Td[(7)]TJF29L963Tf

## 2 C

We shall be using a number of tools from Maple in the work below. The coe cients appearing in the expansions (4) and (5) can be computed from their generating functions as follows. The 2-associated Stirling subset numbers are defined by the generating function

$$(e^z - 1 - z)^m = m! \sum_{n \quad 0} \frac{z^n}{n!} \left\{ {n \atop m} \right\}_{2} :$$

Substituting into the defining equation $We^W = z$, we obtain

$$\left(\ln z - \ln(p + \ln z) + u\right)\frac{ze^u}{p + \ln z} = z$$

From this, it is clear that if we define

$$= \frac{1}{p + \ln z} \text{ and } = \frac{p + \ln(p + \ln z)}{p + \ln z} ; \tag{9}$$

then we recover the equation originally given by de Bruijn for $u$.

$$1 - \quad + \quad u - e^{\quad u} = 0 : \tag{10}$$

The remarkable property is that (10) is invariant with respect to $p$, with only the definitions of and

We wish to investigate first the domains of $z$ $\mathbb{R}$ for which the various series above converge, and how the domains vary with $p$. We begin with a theoretical result for $p = 0$.

**Theorem 1.** *The series (4) converges for $p = 0$ for all $z \geq e$.*

*Proof.* For $p = 0$, we have $= -$ ln

The domain of convergence cannot be extended to $z < e$, because the series for $du{=}dz$ diverges at $z = e$. This can be seen by noting that  $= 0$ at $z = e$ (for $p = 0$). All terms reduce to zero except $m = 1$ which gives the sum

$$\frac{1}{e}\sum_{k=0}^{1}($$

confirms that the point of divergence moves to larger $z$ for decreasing $p$ and to the left for increasing $p$.

**Fig. 3.** For series (5), the ratio $W^{(40)}(z,p)/W(z)$ as functions of $z$ for $p = -1, 0, 1$. Compared with Figure 2, this shows convergence down to smaller $z$.

A similar investigation of series (6) shows an interesting non-monotonic change in the domain of convergence. In Figure 4 the partial sums are plotted and the boundary of the domain of convergence moves to the right for $p = 0$.

We can summarize these findings by noting that series (5) has the widest domain of convergence, and the best behaviour with $p$, while the domains of convergence for series (4) and (6) become worse in that order.

## 5

By rate of convergence, we are referring to the accuracy obtained by partial sums of a series. Given two series, each summed to $N$ terms, the series giving on average a closer approximation to the converged value is said to converge more quickly. The qualification `on average' is needed because it will be seen in the plots below that the error regarded as a function of $z$ can show fine structure which confuses the search for a general trend. Further, the comparison of rate of convergence between different series can vary with $z$ and $p$. For some ranges of $z$, one series will be best, while for other ranges of $z$ a different series will be best. Although one series may converge on a wider domain than another, there is no guarantee that the same series will converge more quickly on the part of the domain they have in common. The practical application of these series is to obtain rapid estimates for $W$ using a small number of terms, and for this the quickest convergence is best, but this will be dependent on the domain of $z$.

1

**Fig. 4.** For series (6), the ratio $W^{(40)}(z,p)/W(z)$ against $z$ for $p = -1, 0, 1, 2$. Compared with figures 2 and 3, the changes in convergence are no longer monotonic in $p$.

The previous section showed that positive values of the parameter $p$ extend the domain of convergence of the series, but its e ect on rate of convergence is di erent. Figures 5, 6 and 7 show the dependence on $z$ of the accuracy of computations of the series (4),(5) and (7) respectively with $N = 10$ for $p = -1; -1=2; 0$ and 1. One can see that the behaviour of the accuracy is non-monotone with respect to both $z$ and $p$ although some particular conclusions can be made. For example, one can observe that for the series (4) at least for $z < 30$ within the common domain of convergence the accuracy for $p = -1=2; 0$ and 1 is higher than for $p = -1$. The series (5) and (7) have the same domain of convergence and a very similar behaviour of the accuracy. Specifically, for these series an increase of positive values of $p$ reduces a rate of convergence within the common domain of convergence i.e. for $z > 1{:}5$. However, at the same time for $z > 11$ computations with $p = -1$ are more accurate than those with positive $p$ and for $5 < z < 18$ the highest accuracy occurs when $p = -1=2$.

The next two figures 8 and 9 display the dependence of convergence properties of the series (4) and (5) respectively on parameter $p$ for di erent numbers of terms $N = 10; 20$ and 40. Again, the curves in these figures confirm that the accuracy strongly depends on parameter $p$ and is non-monotone and show that on the whole an increase of the number of terms improves the accuracy. It is also interesting that there exists a value of $p$ for which the accuracy at the given point is maximum; this value depends very slightly on $N$ and approximately is $p \quad -0{:}75$ in Figure 8 and $p \quad -0{:}5$ in Figure 9.

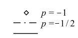


**Fig. 5.** For series (4) with $N = 10$, changes in accuracy in $z$ for $p = -1, -1/2, 0$ and 1.

**Fig. 6.** For series (5) with $N = 10$, changes in accuracy in $z$ for $p = -1, -1/2, 0$ and 1.

**Fig. 7.** For series (7) with $N = 10$, changes in accuracy in $z$ for $p = -1, -1/2, 0$ and 1.

**Fig. 8.** For series (4), the accuracy as a function of $p$ at fixed point $z = 18$ for $N = 10, 20$ and 40.

**Fig. 9.** For series (5), the accuracy as a function of $p$ at fixed point $z = 9$ for $N = 10, 20$ and 40.

## 6 B −1 *p*

The above discussion has considered only real values for the parameter $p$. We briefly shift our consideration to complex $p$ and to branch $-1$. For $z$ in the domain $-1/e < z < 0$, we have that $W\ _1(z)$ takes real values in the range $[-1; -\ )$. The general asymptotic expansion (2) takes the form

$$W\ _1(z) = \ln(z) - 2\ i - \ln(\ln(z) - )2\ i$$

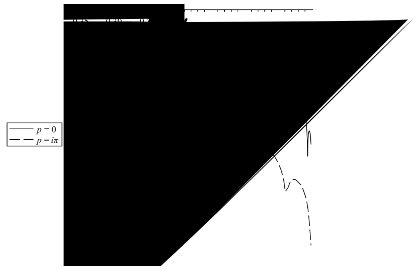


**Fig. 10.** Errors in approximations (18) and (19) for $W_{-1}$.

## 7

We have seen that the transformation allows us to obtain series valid for a wider and the Lagrange Inversion Theorem [11]. To apply this theorem it is convenient to introduce a function that is zero at $t = 0$. We consider the function

$$v = v(t) = W(e^{t})=! - 1 \qquad (26)$$

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