Lagrange inversion and Lambert W

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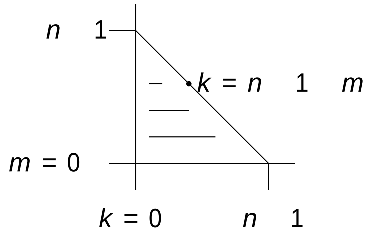
Abstract—**We show that Lagrange inversion can be used to obtain closed-form expressions for a number of series expansions**



This diagram corresponds to the sum

$$\sum_{m=0}^{n-1} \sum_{k=0}^{n-1-m} Q_{nk} \binom{n-1-k}{m} w^m .$$

To obtain the theorem statement, the k and m, which are dummy variables, must be swapped.

$$\sum_{k=0}^{n-1} \sum_{m=0}^{n-1-k} Q_{nm} \binom{n-1-m}{k} w^k .$$

It is now straightforward to equate coefficients of w^k in (18) and obtain the theorem statement (14). The other statements are proved in the same manner. 2

We introduce the next theorem with a general discussion. Given an analytic function $y = f(x)$ and its inverse $x = f(y)$, it is well known that $f(f(y)) = y$ and $f(f(x)) = x$. For Lagrange inversion, both functions are known through series expansions, and we consider a consequence of this. Substituting into (5), we see

$$x = \sum_{k \ge 1} f_k [f(x)]^k .$$

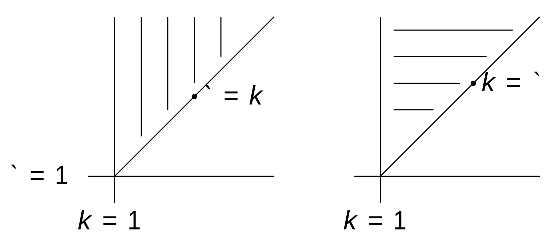
Denoting the expansion of $f(x)^k$ by

$$f(x)^k = \sum_{\ell \ge k} f_\ell^{(k)} x^\ell ,$$

where, since $f(0) = 0$ by assumption, we start the sum at $\ell = k$, we obtain the identity

$$x = \sum_{k \ge 1} f_k \sum_{\ell \ge k} f_\ell^{(k)} x^\ell . \tag{19}$$

To invert the order of summation, we again use a diagram.



The left diagram shows the sum in (19) with the sums over ℓ

At this stage, we still have an infinite series. After the final

We finally re-express the binomial factor in more conventional form:

$$w_n = \sum_{k=0}^{n} 1 \; n$$