Definition 1 (RC and CRC). Let $p(x) \in \mathbb{R}[x]$. The root classification (RC) of $p(x)$ is denoted by $[[n_1, n_2, \ldots], [m_1, -m_1, m_2, -m_2, \ldots]]$ where n_k are the multiplicities of the distinct real roots of $p(x)$, and m_k

2.1. *Discriminant Sequence and Related Sequences*

Yang, Hou & Zeng (1996) defined the following quantities as the basis of their algorithm. Let $p \in \mathbb{R}[x]$ and write $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n \neq 0$.

Definition 2. The discrimination matrix of p is the $2n \times 2n$ matrix

$$M = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \ldots & a_0 & & & & \\ 0 & na_n & (n-1)a_{n-1} & \ldots & a_1 & & & & \\ & a_n & a_{n-1} & \ldots & a_1 & a_0 & & & \\ & 0 & na_n & \ldots & 2a_2 & a_1 & & & \\ & & & \vdots & \vdots & & & & \\ & & & & a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \\ & & & & 0 & na_n & (n-1)a_{n-1} & \ldots & a_1 \end{pmatrix}. \tag{1}$$

Definition 3 (Discriminant Sequence)**.** For $1 \leq k \leq 2n$, let M_k be the kth principal minor of M, and let $D_k = M_{2k}$. The n-tuple $D = [D_1, D_2, \ldots, D_n]$ is called the discriminant sequence of p.

Definition 4 (Sign List)**.** If $[D_1, D_2, \ldots, D_n]$ is the discriminant sequence of p and $\operatorname{sgn} x$ is the signum function with $\operatorname{sgn} 0 = 0$, then the sign list of p is $[\operatorname{sgn} D_1, \operatorname{sgn} D_2, \ldots, \operatorname{sgn} D_n]$.

Definition 5 (Revised Sign List)**.** The revised sign list $[e_1, e_2, \ldots, e_n]$ of p is constructed from the sign list $s = [s_1, s_2, \ldots, s_n]$ of p as follows.

If $[s_i, s_{i+1}, \ldots, s_{i+j}]$ is a section of s, where $s_i \neq 0$, $s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0$ and $s_{i+j} \neq 0$, then we replace the subsection $[s_{i+1}, \ldots, s_{i+j-1}]$ by $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, \ldots]$ such that $e_{i+r} = (-1)^{\lfloor (r+1)/2 \rfloor} s_i$ $(r = 1, 2$]TJ/F39.96Tf6.961.49TD[(6)]TJ/D[(=) -277 (( )]2840T267-1eTJ/F

2.2. Sturm-Habicht Sequence and Related Sequences

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ and $q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$ be two real polynomials with $n = \deg(p) > m = \deg(q)$.

In this section, we introduce the concept of subresultant sequence which comes from Sylvester (1853) and Collins (1967), the concept of Sturm-Habicht sequence which was

Remark 11. The relationship between the discriminant sequence $[D_1; : : : ; D_n]$ of p and the principal Sturm-Habicht coe±cients of p and 1 is: $D_j = a_n$ stha_{n-j}(=

Proposition 14. *Let p 2 R[x] have revised sign list* rsl(p). *If the number of non-vanishing elements in* rsl(p) *is s, and the number of sign changes in* rsl(p) *is v, then $p(x)$ has v pairs of distinct complex conjugate roots and s ¡ $2v$ distinct real roots.*

Proposition 15. *If* ¢^{j}(p) *has k distinct roots with respective multiplicities n_1*

Since each of $D_4; D_5; D_6$ can be positive, zero or negative, there are 27 possible realizations of this sign list to examine. Let us consider the following example sign list: the case $D_4 < 0; D$

mapping © from a sign list to a revised sign list. Therefore the existing algorithms require the inverse mapping $©^{-1}$. However, © is not injective, so $©^{-1}$ is multivalued, and more importantly is difficult to compute.

As an example, consider the polynomial $p_6 := x^6 + ax^2 + bx + c$, whose discriminant sequence was given in (3). One condition (among many) for p_6 having no real roots is that its revised sign list be $[1, -1, -1, 1, -1, -1]$. According to the special structure of the discriminant sequence of p_6, we have

$$©^{-1}[1, -1, -1, 1, -1, -1] = \{[1, 0, 0, 1, -1, -1], [1, 0, 0, 1, 0, -1], [1, 0, 0, 0, -1, -1]\}.$$

Therefore, the given condition is transferred to the following:

$$[D_4 > 0 \wedge D_5 < 0 \wedge D_6 < 0] \vee [D_4 > 0 \wedge D_5 = 0 \wedge D_6 < 0] \vee [D_4 = 0 \wedge D_5 < 0 \wedge D_6 < 0].$$

This case, already cumbersome, is none the less relatively simple because of the nature of the polynomial. However, if the polynomial were a general parametric polynomial, it would be very difficult to find $©^{-1}[1, -1, -1, 1, -1, -1]$, and of course more so for higher degrees. Consequently, it would be a great improvement to avoid revised sign lists.

The second point concerns the realizability of the conditions obtained by the inverse mapping $©^{-1}$. We continue with the example of $x^6 + ax^2 + bx + c$.

Example 18. 6

where $\epsilon_{n-m} = (-1)^{(n-m)(n-m-1)/2}$.

The main theorem for the improved CRC algorithm requires the following lemmas which can be found in Basu, Pollack & Roy (2003). Let $\mathrm{Ind}(q/p)$ be the Cauchy index of q/p on $\mathbb{R}$.

Lemma 21. *Given two polynomials $p(x), q(x)$ in $\mathbb{R}[x]$, we have $\mathrm{TaQ}(q, p) = \mathrm{Ind}(p'q/p)$.*

Lemma 22. *Let $p(x), q(x)$ be the two polynomials in Section 2.2. We have*

$$\mathrm{PmV}([\mathrm{sRes}_n(p, q), \mathrm{sRes}_{n-1}(p, q), \ldots, \mathrm{sRes}_0(p, q)]) = \mathrm{Ind}(q/p) .$$

The main theorem is the following

Theorem 23. *Let $D = [D_1, \ldots, D_n]$ be the discriminant sequence of a real polynomial $p(x)$ of degree n, and let ℓ be the maximal subscript such that $D_\ell \neq 0$. If $\mathrm{PmV}(D) = r$, then $p(x)$ has $r + 1$ distinct real roots and $\frac{1}{2}(\ell - r - 1)$ pairs of distinct complex conjugate roots.*

Proof. We first prove that

$$\#\{\alpha \in \mathbb{R} | p(\alpha) = 0\} = \mathrm{PmV}([D_1, \ldots, D_n]) + 1 .$$

Observe that $\#\{\alpha \in \mathbb{R} | p(\alpha) = 0\} = \mathrm{TaQ}(1, p)$. Then from Lemma 21, we have $\mathrm{TaQ}(1, p) = \mathrm{Ind}(p'/p)$. By Lemma 22,

$$\mathrm{Ind}(p'/p) = \mathrm{PmV}([\mathrm{sRes}_n(p, p'), \mathrm{sRes}_{n-1}(p, p'), \ldots, \mathrm{sRes}_0(p, p')]) .$$

By Remark 13,

$$\begin{aligned} &\mathrm{PmV}([\mathrm{sRes}_n(p, p'), \mathrm{sRes}_{n-1}(p, p'), \ldots, \mathrm{sRes}_0(p, p')]) \\ &\quad = \mathrm{PmV}([\mathrm{sgn}(a_n), D_1/a_n, \ldots, D_n/a_n]) = 1 + \mathrm{PmV}([D_1/a_n, \ldots, D_n/a_n]) , \end{aligned}$$

since $\mathrm{sgn}(a_n)$ and $D_1/a_n = na_n$ have the same sign. Finally,

$$1 + \mathrm{PmV}([D_1/a_n, \ldots, D_n/a_n]) = 1 + \mathrm{PmV}([D_1, \ldots, D_n])$$

Example 25. We give an example of the use of the above corollary, by proving the non-realizability of condition (4) from a different point of view. The condition is equivalent to the sign list $[1, 0, 0, 0, -1, -1]$, which has revised sign list $[1, -1, -1, 1, -1, -1]$. Since

$$\mathrm{PmV}([1, 0, 0, 0, -1, -$$

Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be a real parametric polynomial with $a_n \neq 0$. The algorithm starts from generating all possible RCs for $p(x)$ using `AllListsReal`. Then for each RC L, we find the conditions on the parametric coefficients of $p(x)$ such that L is realized.

We first compute all possible sign lists of $p(x)$ for $p(x)$ having L as its RC.

Algorithm 1. GenAllSL
Input: A real parametric polynomial $p(x)$ and an RC L.
Output: The set of all the sign lists of $p(x)$ that lead to the RC given by L.
Procedure:

- $[n, \ell, r] \leftarrow$ `RCInfo`(L).
- Compute the discriminant sequence $D = [D_1, \ldots, D_n]$ of p.
- Compute the set S_0 of all possible sign lists from D: for $1 < k \leq n$, if $D_k \in \mathbb{R}$, then $D_k \to \mathrm{sgn}(D_k)$; otherwise, $D_k \to \{-1, 0, 1\}$. For example, if $D = [1, -2, a]$, then $S_0 = \{[1, -1, -1], [1, -1, 0], [1, -1, 1]\}$.
- Compute $S = \{s \in S_0 \mid$ `MaxSubs`$(s) = \ell, \mathrm{PmV}(s) = \mathrm{PmV}(\mathrm{rsl}(s)) = r - 1\}$,
- Return S.

Then $S =$ `GenAllSL`(p, L) is the set of all possible sign lists of $p(x)$ for $p(x)$ having L as its RC. In order to make the multiplicities of the roots of $p(x)$ be those specified by L, we also have to determine the possible sign lists of the polynomials in the Δ-sequence of $p(x)$ (Definition 6), except for the following five cases (*termination conditions*): if the RC of $p(x)$ is L and is such that $[n, \ell, r] =$ `RCInfo`(L), then these cases are

(1) $n = \ell$,
(2) $\ell = 1$.
(3) $\ell = 2$ and $r = 0$.
(4) $n - \ell = 1$.
(5) $r = 0$ and $n - \ell = 2$.

For other cases, $\Delta^1(p) = \Theta_{n-\ell}(p)$, the $(n - \ell)$th multiple factor of $p(x)$ (Definition 7). By Proposition 15, the RC of $\Delta^1(p)$ would be $L_1 =$ `MinusOne`(L). Then we can call `GenAllSL` recursively. This is the basis of the following algorithm which generates the conditions for $p(x)$ having L as its root classification. The output conditions are a sequence of *mixed lists*. Each mixed list consists of a polynomial in the Δ-sequence of $p(x)$, followed by all of its possible sign lists. We denote the empty sequence by NULL. Notice that if NULL is returned, then L is not realizable.

Algorithm 2. Cond
Input: a real parametric polynomial $p(x)$; an RC L.
Output: A sequence of mixed lists (the conditions for $p(x)$ having L as its RC).
Procedure:

$[n, \ell, r] \leftarrow$ `RCInfo`(L)
$S \leftarrow$ `GenAllSL`(p, L)
if $S = \emptyset$
    return NULL
else if $[n, \ell, r]$ meets one of the five cases
    return $[p,$ `Op`$(S)]$
else

C $\tilde{A}$ Cond

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Table 1. Numbers of Non-realizable Sign Lists Detected by Corollary 24

Degree n	2	3	4	5	6	7	8	9	10	11
3^{n-1}	3	9	27	81	243	729	2187	6561	19683	59049
Detected										

```
      [p6, [1,0,0,1,0,0]]
(10) [[],[1,-1,1,-1,1,-1]], if and only if
      [p6, [1,0,0,0,1,-1],[1,0,0,-1,1,-1],[1,0,0,1,0,-1],
        [1,0,0,0,0,-1],[1,0,0,1,-1,-1],[1,0,0,1,1,-1]]
where
(#1) p6:=x^6+a*x^2+b*x+c,
and its discriminant sequence is:
[1, 0, 0, a^3, 256*a^5+1728*c^2*a^2-5400*a*c*b^2+1875*b^4,
-1024*a^6*c+256*a^5*b^2-13824*c^3*a^3+43200*c^2*a^2*b^2
-22500*b^4*c*a+3125*b^6-46656*c^5]
(#2) p62:=4*a*x^2+5*b*x+6*c,
and its discriminant sequence is:
 [1, 25*b^2-96*a*c]
```

Let us explain the CRC of p_6 with respect to the improved algorithm. First, the algorithm CRC calls the function AllListsReal to generate all possible root classi¯cations (RCs) for a polynomial of degree 6. There are 23 RCs as follows. For the sake of simplicity, the order of them has been changed.

```
[ [[3,3],[]], [[2,4],[]], [[2,2,2],[]], [[1,5],[]], [[1, 2,3],[]],
[[1,1,4],[]], [[1,1,2,2],[]], [[1,1,1,3],[]], [[1,1,1,1,2],[]],
[[1,1,1,1,1,1],[]], [[4],[1,-1]], [[2],[2,-2]], [[],[3,-3]],
[[6],[]], [[1,1,1,1],[1,-1]], [[1,1,2],[1,-1]], [[1,3],[1,-1]],
[[2,2],[1,-1]], [[1,1],[2,-2]], [[1,1],[1,-1,1,-1]], [[2],
[1,-1,1,-1]], [[],[1,-1,2,-2]], [[],[1,-1,1,-1,1,-1]] ]
```

Second, in a \for-loop", for each RC L above and p_6, the algorithm Cond is called to generate the conditions for p_6

lists of p_6 for p_6 having L as its RC, and it turns out that $S = \{[1, 0, 0, -1, 0, 0]\}$. Because $S \neq \emptyset$ and $[n, \ell, r]$ does not meet the termination conditions, Cond also has to compute all possible sign lists of ¢$^{1}(p_6)$ which is p

```
      [p6, [1,0,-1,0,0,1],[1,0,0,0,0,1],[1,0,-1,-1,0,1],
        [1,0,0,-1,0,1],[1,0,-1,-1,-1,1],[1,0,0,-1,-1,1],
        [1,0,-1,1,1,1],[1,0,0,1,1,1],[1,0,-1,0,1,1],
        [1,0,0,0,1,1],[1,0,-1,-1,1,1],[1,0,0,-1,1,1]]
(9) [[2],[2,-2]], if and only if
      [p6, [1,0,-1,0,0,0]], [p63, [1,1,-1],[1,0,-1],[1,-1,-1]]
(10) [[2],[1,-1,1,-1]], if and only if
      [p6, [1,0,-1,1,1,0],[1,0,0,1,1,0],[1,0,-1,0,1,0],
        [1,0,0,0,1,0],[1,0,-1,-1,1,0],[1,0,0,-1,1,0]]
(11) [[],[1,-1,2,-2]], if and only if
      [p6, [1,0,0,1,0,0],[1,0,-1,1,0,0]]
(12) [[],[1,-1,1,-1,1,-1]], if and only if
      [p6, [1,0,0,1,0,-1],[1,0,-1,0,0,-1],[1,0,0,0,0,-1],
        [1,0,-1,1,-1,-1],[1,0,0,1,-1,-1],[1,0,-1,1,1,-1],
        [1,0,0,1,1,-1],[1,0,-1,0,1,-1],[1,0,0,0,1,-1],
        [1,0,-1,-1,1,-1],[1,0,0,-1,1,-1],[1,0,-1,1,0,-1]]

where
(#1) p62:=-9*c*a^3-180*d*c*a+192*d*b^2+Q1*x+Q2*x^2,
(#2) p6:=x^6+a*x^3+b*x^2+c*x+d,
(#3) p63:=-3*a*x^3-4*b*x^2-5*c*x-6*d,
 and
Q1:=160*c*b^2-18*b*a^3-150*a*c^2-144*a*d*b,
Q2:=-27*a^4+108*d*a^2-240*a*b*c+128*b^3,
```

$[1,0,0,1,-1,-1] \iff [-a^2 = 0 \wedge D_4 > 0 \wedge D_5 < 0 \wedge D_6 < 0]$
$[1,0,-1,1,1,-1] \iff [-a^2 < 0 \wedge D_4 > 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1,0,0,1,1,-1] \iff [-a^2 = 0 \wedge D_4 > 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1,0,-1,0,1,-1] \iff [-a^2 < 0 \wedge D_4 = 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1,0,0,0,1,-1] \iff [-a^2 = 0 \wedge D_4 = 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1,0,-1,-1,1,-1] \iff [-a^2 < 0 \wedge D_4 < 0 \wedge D_5 > 0 \wedge D_6 < 0]$
$[1,0,0,-1,1,-1] \iff [-a^2 = 0 \wedge D_4 < 0 \wedge D_5 > 0 \wedge D_6 < 0]$ D;[1†
$[1,0,-1,1,0,-1] \iff [-a^2 < 0 \wedge D_4 > 0 \wedge D_5 = 0 \wedge D_6 < 0]$

Simplifying by hand or by QEPCAD (Brown, 2004), we conclude that case (12) holds $\iff D_6 < 0 \wedge [D_4 > 0 \vee D_5 > 0 \vee [D_4 = 0 \wedge D_5 = 0]]$.

Finally, by combining the conditions for case (11) and case (12), we obtain the desired result: $(\forall x)[p_6 > 0] \iff [D_4 > 0 \wedge D_5 = 0 \wedge D_6 = 0] \vee [D_4 = 0 \wedge D_5 = 0 \wedge D_6 < 0] \vee [D_4 > 0 \wedge D_6 < 0] \vee [D_5 > 0 \wedge D_6 < 0]$.

Example 29. Find the conditions on a,b,c,d such that $(\forall x)[x^8 + ax^3 + bx^2 + cx + d > 0]$.

$[D_4$+[D[D

7. Conclusion

In this paper, we have proposed an improved algorithm for the automatic computation of the complete root classi¯cation of a real parametric polynomial, and a new test for non-realizable conditions. However, some issues deserve further consideration. For example,

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