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to all orders. In the appendix by Van Dyke, two dimensional problems which

points the mistake however of writing an infinite sum where now we write a  $\Pi$ -product

and  $Q_d = 0$ . An attempt to calculate  $Q_b^{(1)}$  from (3.3b) by this method leads to an integrand  $e^{-2s} \coth s$  which would give an infinite value for  $Q_b^{(1)}$ . Obviously the approximation  $\delta^{(1)}$  breaks down in the neighborhood of the pole where its gradient

$$\phi^{(2)} = \frac{1}{2} \tau^3 (2 - H) / H^3 - \frac{1}{2} \tau (H^2 - 2H + 2) / H^2 \quad (4.3b)$$

To calculate the quantities defined in Section 2, we shall need  $\partial\phi/\partial n$  on the sphere surface within the gap; this is most easily expressed in terms of  $Z$ :

$$\frac{\partial\phi}{\partial n} = \epsilon^{-1} \frac{\partial\phi}{\partial Z} + (1 - Z) \frac{\partial\phi}{\partial r} - R \frac{\partial\phi}{\partial \theta} + O(\epsilon)$$

$$= \epsilon^{-1} / Z + \frac{1}{3} (1 + 1/Z) + O(\epsilon). \quad (4.4)$$

##### 5 The Second-order Solution Outside the Cap.

result

$$\eta^2 \int_0^x f(s) J_0(s\eta) ds = \int_0^x [(f/s)' - f''] J_0(s\eta) ds, \quad (5.1)$$

which is true provided  $f' - f/s \rightarrow 0$  as  $s \rightarrow 0$ . The prime denotes  $d/ds$ . For problem (c) the boundary condition becomes

$$\phi_c^{(2)}(1, \eta) = -(1 + \eta^2)^{\frac{1}{2}} \int_0^x C(s) \sinh s J_0(s\eta) ds,$$

where

$$C(s) \sinh s = e^{-s} - 3s e^{-s} + 2s^2 e^{-s} + s^2 e^{-s} \coth s - (s e^{-s} \coth s)' + (s^2 e^{-s} \coth s)''.$$

The required solution for  $\phi_c^{(2)}$  is then obviously

$$\phi_c^{(2)}(\xi, \eta) = -(\xi^2 + \eta^2)^{\frac{1}{2}} \int_0^x C(s) \sinh s \xi J_0(s\eta) ds.$$

For problem (b) things are not so straightforward. Before we can use (5.1) we must separate the  $s^{-1}$  singularity in  $\coth s$ ; we obtain

$$\phi_b^{(2)}(1, \eta) = -\frac{1}{2}(1 + \eta^2) + (1 + \eta^2)^{\frac{1}{2}} \int_0^\infty B(s) \sinh s J_0(s\eta) ds, \quad (5.2)$$

where

$$B(s) \sinh s = s e^{-s} + \frac{1}{2} s e^{-s} \coth s - \frac{1}{2} (e^{-s} \coth s - e^{-s}/s)' + \frac{1}{2} (s e^{-s} \coth s - e^{-s})''.$$

We must now find a particular integral of Laplace's equation that will fit this boundary condition. We find it by substituting

$$W(\xi, \eta) = \int_0^\infty \xi(\xi^2 + \eta^2)^{-\frac{1}{2}} \eta J_0(s\eta) d\eta,$$

into Laplace's equation, taking an inverse Hankel transform and obtaining an equation for  $W$ :

$$\partial^2 W / \partial \xi^2 - s^2 W + 4 \int_0^\infty \xi(\xi^2 + \eta^2)^{-\frac{1}{2}} \eta J_0(s\eta) d\eta = 0.$$

A particular solution of this equation that satisfies boundary condition (5.2) is

We now proceed to calculate  $Q$ ,  $S$  and  $F$  and compare the expressions with other work.

### 6. Calculation of $Q$

The quantity  $Q^{(1)}$  was calculated in Section 3. We start this section by calculating

$Q_c^{(2)}$ , which comes easily from  $\phi_c^{(2)}$ . We must first express our definition (2.1) in terms of the  $(\xi, \eta, \theta)$  coordinates we have been using. At first sight it may seem that this requires expressions for  $\partial\phi/\partial n$  and  $dA$  on  $\xi = 1 + \frac{1}{2}c(\eta^2 - 1)$ , but  $Q$  is independent of the surface chosen, provided it surrounds only one sphere, and so we choose the surface  $\xi = 1$ . Then

$$Q = \int \left[ \frac{\partial\phi}{\partial\xi} \right]_{\xi=1} \frac{2\eta d\eta}{1+\eta^2}.$$

The calculation of  $Q_c^{(2)}$  follows exactly that of  $Q_c^{(1)}$ , giving

where all integrals can be evaluated using integration by parts and Gradshteyn & Ryzhik (1965, Section 3.552).



$$Q_{bo} = T_b \ln Z_0 + \frac{1}{3} T_b (\ln Z_0 + Z_0 + 1) + O(\epsilon^2).$$

Substituting for  $Z_0$  gives

$$Q_{bo} = T_b [\ln 2 - \ln \epsilon - \ln(1 + \eta_0^2)] + \frac{1}{3} T_b \left[ \frac{1}{2} \ln 2 - \frac{1}{2} \ln \epsilon - \frac{1}{2} \ln(1 + \eta_0^2) + \frac{1}{2} \eta_0^2 - \frac{3}{2} + O(\eta_0^2) + O(\eta_0^{-1}) \right]$$

Adding  $Q_{bi}$  and  $Q_{bo}$  cancels the singular terms and gives

We now compare this result with those derived by Keller (1963) and Batchelor &

which, as stated, contains no singular terms. Thus, combining this with the fact that only the  $z$  component of  $\mathbf{S}$  is non-zero, we can reduce our expression for  $S$  to

$$S = 2\pi a^2 \int_0^z \left\{ \left[ \frac{\partial \phi}{\partial \zeta} \right]_{\zeta=1} - \varepsilon(1-z)\phi^{(1)} \right\} \frac{4\eta d\eta}{(1+\eta^2)^2} + O(\varepsilon^2).$$

Performing the integrations gives (using  $\zeta$  for the Riemann zeta function)

$$S_{bc} = \frac{4}{3}\pi a^2 T_{bc} \left[ \frac{1}{2}\pi^2 + \varepsilon \left( \frac{1}{3}\pi^2 - \frac{1}{2} \right) \right] - \frac{4}{3}\pi a^2 G_c [6\zeta(3) + \varepsilon(4\zeta(3) - 3\zeta(4))], \quad (7.1)$$

where  $T_{bc}$  is given by (6.1), and  $S_d^{(1)} = 3\pi a^2 \zeta(3) G_d$ . Smith & Rungis (1975) and Love (1975) obtained the leading-order  $G_c$  term for  $S_{bc}$  but not the  $T_{bc}$  term.

The expressions with which we shall compare these results come from Jeffrey (1973) and are exact for all separations; in the present notation,

$$S_{bc} = \frac{4}{3}\pi a^2 G_c \sum_{p=0}^{\infty} A_{01p} t^p \quad \text{and} \quad S_d = \frac{4}{3}\pi a^2 G_d \sum_{p=0}^{\infty} A_{11p} t^p,$$

where  $t = \frac{1}{2}(1+\varepsilon)^{-1}$  and the coefficients are given by

$$A_{mn0} = 3\delta_{1n} \quad \text{and} \quad A_{mnp} = (-1)^m \sum_{s=1}^{(p-n-3)/2} \binom{n+s}{n+m} A_{ms(p-n-s-1)}.$$

We wish to test the rate at which the infinite series converge, both to test their

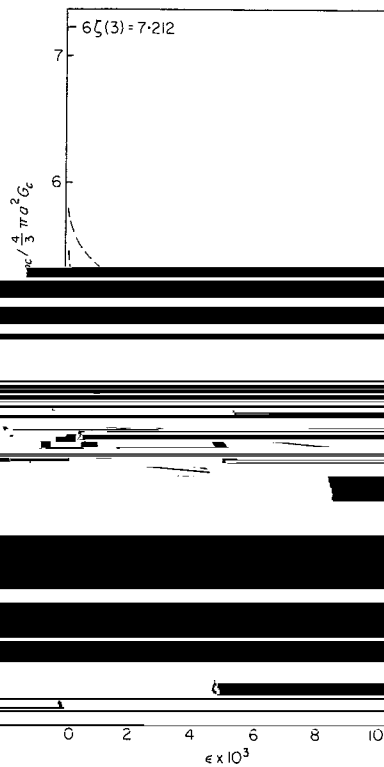


FIG. 2. Comparison of different expressions for  $S_{bc}$ . The broken lines actually start from 7.212 at  $\epsilon = 0$ . The curves are: (1)  $6\zeta(3) + \frac{1}{8}\pi^4/\ln \epsilon$ ; (2)  $6\zeta(3) + \frac{1}{8}\pi^4/(\ln \epsilon - \ln 2 - 2\gamma)$ ; (3) the series summed to  $P = 70$ ; (4) the series summed to  $P = 150$ .

$F_{bc}$ , which has not yet been studied. Warren & Cuthrell (1975) have calculated  $F_b$  and also measured it experimentally; Davis (1964) has calculated  $F_{bc}$ ; Smith & Barakat (1975) have calculated  $F_c$ . Since  $T_{bc} \rightarrow 0$  as  $\epsilon \rightarrow 0$ , one might expect that  $F_{bc} \rightarrow F_c$  as  $\epsilon \rightarrow 0$ . The numerical results of Davis (1964) and Smith & Barakat (1975) throw doubt on this and here we shall see explicitly that it is not true. In fact  $F_{bc} \rightarrow \infty$  as  $\epsilon \rightarrow 0$ .

Only the  $z$  components of  $F_b$ ,  $F_c$  and  $F_{bc}$  are non-zero (neglecting possible complications produced by problem (d)). Squaring (4.4) and integrating appropriately gives the contribution from the gap as

$$F_{bo} = 2\pi T_b^2 \left\{ \epsilon^{-1} (1 - 1/Z_0) - \frac{1}{3} \ln Z_0 + \frac{5}{3} (1 - 1/Z_0) \right\} + O(\epsilon) \\ = 2\pi T_b^2 \left\{ \epsilon^{-1} + \frac{1}{3} \ln \epsilon + \frac{5}{3} - \frac{1}{3} \ln 2 - \frac{1}{2} (1 + \eta_0^2) + \frac{1}{3} \ln (1 + \eta_0^2) \right\} + \\ O(\epsilon) + O(\eta_0^{-2}).$$

Note that both terms of the gap solution are needed to cancel the singularities in the first term of the solution outside the gap. The contribution of (3.3b) to the force is

$$F_{bi}^{(1)} = 2\pi T_b^2 \int_0^{\eta_0} \left[ \frac{\partial \phi_b^{(1)}}{\partial \xi} \right]_{\xi=1}^2 \frac{\eta^2 - 1}{\eta^2 + 1} \eta d\eta. \tag{8.1}$$

We first separate those terms in  $\partial \phi / \partial \xi$  which will lead to singular terms in  $\eta_0$ . Thus

$$\left[ \frac{\partial \phi_b^{(1)}}{\partial \xi} \right]_{\xi=1} = 1 + \frac{2/3}{1 + \eta^2} + (1 + \eta^2)^{\frac{1}{2}} \int_0^\infty s e^{-s} \left( \coth s - \frac{1}{s} + \frac{1}{3} \right) J_0(s\eta) ds.$$

Separating this into singular and non-singular terms (8.1) gives

It will be seen below that the singular terms alone in  $F_b$  will be sufficient to reproduce the numerical results of Davis (1964) and Wapner & Cuthrell (1975) and so the  $O(1)$

$$F_b^{(1)} = 2\pi T_b^2(\varepsilon^{-1} + \frac{1}{3} \ln \varepsilon + O(1)). \quad (8.2)$$

We compare (8.2) first with the results of Wapner & Cuthrell (1975) who solved

~~Figure 2. Almost Touching Circles~~

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to our replaces the spheres of the main paper by parallel, almost touching circles

cylinders, the various approximate solutions to problems (b)–(d) can be found in simple closed form. There is no counterpart of problem (a), for if two circles are at equal temperatures there is a logarithmic singularity in the potential at infinity unless the temperature is constant everywhere.

$$H = 1 + \frac{1}{2}Y^2 + \frac{1}{2}\epsilon^2 Y^4 + \frac{1}{2}\epsilon^4 Y^6 + \dots \quad (\text{A.6})$$

where  $H = 1 + \frac{1}{2}Y^2$  as one would expect. The first term is the solution of Keller (1963). Although we have made no use of the boundary condition far from the circles, this matches with the solution there (i.e. this outer approximation matches the inner approximation (A.1)). That is, rewriting the one-term solution (A.1) in the gap variables  $X, Y$ , expanding for small  $\epsilon$  and keeping two terms gives  $2X/Y^2 - 2\epsilon X^3/Y^4$ , conversely, rewriting the two-term solution (A.6) in the variables used outside the gap  $x, y$ , expanding for small  $\epsilon$  and keeping one term gives  $2x/y^2 - 2x^3/y^4$ ; and these are the same.

This might suggest that in this singular perturbation problem matching has played no role at this stage; but that is not true. The solution (A.1) is in fact not unique, for we can add to it any multiple of any of the "trigonometric solutions"

$$\sin N\pi\xi \cosh N\pi\eta, \quad N = 1, 2, 3, \dots \quad (\text{A.7})$$

periodic array of cylinders). Thus the singularities in the last term of (A.10) must also cancel. To verify this, we continue the solution outside the gap in the form  $\phi^{(1)} + \varepsilon\phi^{(2)} + \dots$

$$\phi_b^{(2)} = \frac{1}{6} \operatorname{Re} (2\zeta + \zeta^3) = \frac{1}{3}\zeta + \frac{1}{6}(\zeta^3 - 3\zeta\eta^2). \quad (\text{A.11})$$

This shows a typical symptom of non-uniformity, being singular like the inverse cube

