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ABSTRACT

We study discrete dynamical systems, or recurrence relations, of the general form

$$y_{n+1} = y_n (y_n)$$

the fixed point is attractive, and $|F'(x)| < 1$, for which it is repelling, are more common, but easier to understand.

We associate with the discrete system an interpolating continuous system $Y(t)$, such that $Y(n) = y_n$. The asymptotic behaviour of y_n can then be estimated through $Y(t)$. A theorem of Poincaré states that the corresponding continuous system is

$$Y'(t) = G(Y(t)), \quad (1)$$

where G is termed the generator, and is given by

$$G(y) = \frac{y_1 y^2}{1 + y_1 y + y_2 y^2 + \dots}. \quad (2)$$

The y_n are related to the P_n by the recurrence relation

$$y_n = \frac{P_{n+1}(y)}{P_n(y)}, \quad (3)$$

$$= \frac{1}{n+1} \sum_{i=1}^n \left[P_{i+1}(y) - \frac{(n+1)P_{i+1}(y)(n-i)}{n+1-i} \right] - i. \quad (4)$$

here

More such identities can be found in
 The expansion of logarithm is thus

$$\begin{aligned} \ln Y &\sim \ln \left(Z + \sum_{\geq 2} c Z \right) && () \\ &\sim \ln Z + \ln \left(1 + \sum_{\geq 2} c Z^{-1} \right) - i\mathcal{K}_0 && () \\ &\sim \ln Z + \sum_{\geq 1} d Z^{-1} - i\mathcal{K}_0, && () \end{aligned}$$

here $\mathcal{K}_0 = \mathcal{K}(\ln Z + \ln(1 + \sum_{\geq 2} c Z^{-1}))$. The d are easily computed functions of the c by elementary algorithms for the computation of the logarithm of a series. Similarly, by reciprocation of a series, we can obtain

$$\bar{Y} \sim \bar{Z} \left(1 + \sum_{\geq 1} e Z^{-1} \right)$$

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— [REDACTED]

Ignore the higher order term and substitute $f(z) = Z$ into (5) and use the Maple command `as mpt` to calculate the asymptotic series of $f(z)$. Surprisingly, we get

$$f(z) = -\frac{1}{C(z)} + O\left(\frac{1}{n}\right) \quad (5)$$

That means

$$f(z) = \lim_{n \rightarrow \infty} f_n(z) = -\frac{1}{C(z)} \quad (6)$$

This gives a new approach for computing $f(z)$.

Use the Lambert W function as an example to compare this new approach to the straightforward approach. We call the straightforward approach `method 1` and our new approach `method 2`. We compute (7) by each method.

- $n = 10$, $f(10) = 0.1051281492$

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Method

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As we can see, method 2 has the advantage of converging faster than method 1.

7. CONCLUDING REMARKS

It has long been known in the analysis of algorithms community that using the Lambert function (also called sometimes the *omega* function, prior to its name being standardized) as useful in economics in some asymptotic analyses.

This present paper shows an example class of problems for which the economic (natural) (admission) (admission) (recon) (y) (tt) (h) (problems) (re)