The Solution of $S\exp(S$

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ABSTRACT

We study the solutions of the matrix equation exp( ) = A. Our motivation comes from the study of systems of delay differential equations (

ferentiable function ( ) such that

$$( ) = A (- 1)$$

where A is an -by- matrix of complex numbers, and ( ) is specified on an initial vector history

$$( ) = ( ) = [_1( ) \quad _2( ) \quad _n( )]^T$$

on the interval $-1 \leq \quad \leq 0$.

As in [7], this is a special problem, useful for some models in mathematical biology and elsewhere. Powerful numerical techniques exist for solving general delay di erential equations; see for example [14]. The ultimate aim of the present work, in contrast, is to look for special-purpose techniques that may be more e cient for these 'niche' problems, or give greater insight. The approach used here is to note that the ansatz

$$( ) = \exp(\quad)$$

for some constant -vector (later to be used as one term in a Fourier-like series solution of the delay di erential equation) leads to some interesting matrix computations, such as the computation of any and all such that[1]

$$\exp(\quad) = A \tag{1.1}$$

We shall consider this equation for $A \quad \in \mathbb{C}^{n\times n}$.

Equation (1.1) is a matrix analogue of the scalar equation $s^{s} = \quad \in \mathbb{C}$, whose solutions are $s = \quad _k( )$, where $_k$ is the th branch of the Lambert function [3]. Ma-

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the lessons learned in computing the matrix logarithm will be of use in solving (1.1) [2]. Here, we will barely scratch the surface of numerical computation for this problem, concentrating mainly on the theoretical aspects, and we will return to the problem of numerical computation in a future paper.

2. MATRI F NCTIONS AND EQ ATIONS

We first recall some definitions. For a more comprehensive discussion, see [9].

If

$$F(\,) = \sum_{k \geq 0} {}_{k}{}^{k}$$

is a convergent power series in $|\,|$, then the $F : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ is defined to be

$$F(A) = \sum_{k \geq 0} {}_{k} A^{k} \tag{2.1}$$

which converges for (A) , where is the spectral radius.

Recalling that any square matrix A has the Schur decomposition $A = \; T$, where is unitary and the diagonal entries of T

case the delay di erential equation reduces to the ordinary di erential equation ( ) = 0 and is of little further interest. Some interest remains in this case when 1, however, as we shall see.

4. SOME GENERAL RES LTS

We are concerned from now on to discover the relations between the matrix function $_k(A)$ defined above, and the solutions of equation (1.1). As with the functional definition (2.1), we can limit our discussion to upper triangular matrices A.

LEMMA 1. A

Proof. If exp( ) = A, and A = $^{-1}$ where is upper triangular, then $^{-1}$ $^{-1}$ exp( ) = and if we put = $^{-1}$ we see that exp( ) = , and thus any solution of the original equation is similar to a solution of the same equation where the input is upper triangular. Conversely, any solution of the upper triangular equation gives a solution = $^{-1}$ of the original equation. □

Henceforth we assume that A is upper triangular. One question that we can then ask is: if exp( ) = A and A is upper triangular must also be upper triangular? We first establish another lemma.

LEMMA 2. exp( ) = A A

Proof

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5. THE T O-B -T O CASE

We now specialize to the case = 2 in order to glean some more insight.

We consider first the case where A is derogatory, which for = 2 implies $A = \lambda I$ for some $\lambda \in \mathbb{C}$. Theorem 2 shows that there are infinitely many diagonalizable solutions to (1.1) for $\lambda \neq 0$. But these are not the only solutions. For example, if $A = \text{diag}(- ^{-1} \; - ^{-1})$ then an easy computation shows that the non-diagonalizable matrix

$$J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

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satisfies J)espr

Finally, we note that if is not constrained to be upper triangular, then the problem of taking the matrix exponential (when the entries of are symbols) as part of any procedure for solving exp( ) = A becomes rather complicated. Consider the 5×5 example at the end of Section 4; even with such a modest example, the nonlinear equations that arise in exp( ) = A are daunting for hand calculation, and the complexity grows rapidly enough with dimension that symbolic methods are in all likelihood not going to be useful for dimensions much larger than 5. Special-purpose numerical schemes for take90.00-415.3(5)9.4(.)]25.1(o)-25.1(d)]TJ T* 0.0TD 0 Tc ak T* (the)aD 0 Tc 18.9(u)7.2(27o)-2(n)1

points; continua of solutions in the derogatory case; and vanishingly small impact of the roots of $\mu \exp \mu = \lambda$ when the eigenvalues λ of A are small.[2]

These di culties should be borne in mind when inves-