A Sequence of Series for The Lambert W Function

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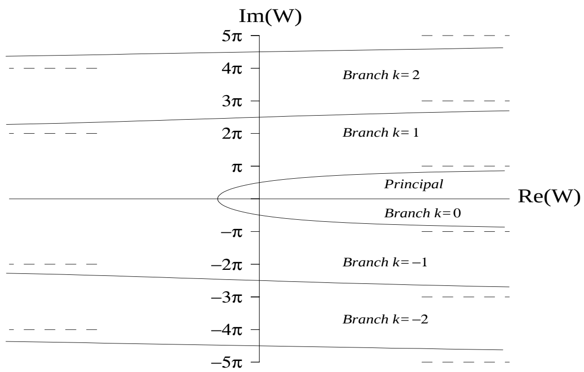

Figure 1: The ranges for the branches of the W function. Closure information is not indicated on the figure. Branches are closed on their top boundaries. This means in particu-

2.1 Some series

As before, when $w = W(\exp a)$ is rational the Taylor series for $W(\exp\ )$ about $\ = a$ has rational coefficients. For example, this occurs when $a = 1$, giving

$$W(e^z) = 1 + \frac{1}{2}(\ -1) + \frac{1}{16}(\ -1)^2 - \frac{1}{192}(\ -1)^3 - \frac{1}{3072}(\ -1)^4 + \frac{13}{61440}(\ -1)^5 + O((\ -1)^6)\,. \qquad (28)$$

This series has radius of convergence $\sqrt{4+\pi^2}$, which is the distance to the nearest singularity at $\ = -1 + i\pi$. This gives the asymptotics of $q_n(1)$.

2. Branches in equation (24)

Theorem: The unique solution of $y + \ln y = \ $ is

$$y = W_{-\mathcal{K}(z)}(e^z) \qquad (29)$$

where $\mathcal{K}(\ )$ is the unwinding number of $\ $ (see [6]), unless $\ = t + i\pi$ for $-\infty < t \le -1$, in which case there are exactly two solutions, $y = W_{-1}(\exp\ )$ and $y = W_0(\exp\ )$.

Proof. Taking exponentials of both sides of $y + \ln y = \ $ we see that if y is a solution, then $y = W_k(\exp\ )$ for some k. To go in the other direction, we use the relation

$$W_k(\ ) + \ln W_k(\ ) = \ln\ + 2\pi i k \qquad (30)$$

unless $k = -1$ and $\ \in [-1/e, 0)$, when $W_{-1}(\ ) + \ln W_{-1}(\ ) = \ln\ $. For a proof of this relation see [12]. We replace $\ $ in the above by $\exp\ $, and since

$$\ln e^z = \ + 2\pi i\mathcal{K}(\ ) \qquad (31)$$

(indeed this defines the unwinding number $\mathcal{K}(\ )$, see [6]), we have that

$$W_k(e^z) + \ln W_k(e^z) = \ + 2\pi i\,(\mathcal{K}(\ ) + k) \qquad (32)$$

unless $k = -1$ and $\exp\ \in [-1/e, 0)$, in which case we replace k by 0 on the right hand side of (32). Thus we have that $W_k(\exp\ ) + \ln W_k(\exp\ ) = \ $ if and only if $k = -\mathcal{K}(\ )$, unless $k = -1$, as claimed. It is easy to see that both $k = -1$ and $k = 0$ work if $\ = t + i\pi$ for some $t \in (-\infty, -1]$ as claimed, and because the unwinding numbers for $\ = t + i(2m+1)\pi$ are all different from 0 if $m \ne 0$, this half-line is the only set of exceptions.

Remark

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> op({solve(series(subs(w=-1+u, z=-exp(-1)
+ t^2* exp(-1),w*exp(w)-z),u),u)}) ;
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$$\sqrt{2}\,t - \frac{2}{3}\,t^2 + \frac{11}{36}$$

for the root that does not simplify. Using the differential equation

$$\mu \frac{d\mu}{}$$

These series were further developed and rearranged in [11] using the new variable $\zeta = 1/(1+\sigma)$ to get

W

In the case $n \to \infty$, one can show that $v_n \to W(\ )$ if $|W(\ )| > 1$, while $v_n \to W(\ )$ in $|W(\ )| < 1$ if $n \to -\infty$.

In fact this iteration turns out to be completely equivalent to the well-studied exponential iteration defining

$$z^{z^{\cdot^{\cdot^{\cdot}}}} \tag{78}$$

and this connection is discussed in detail elsewhere [7]. For our purposes we note that this gives us an infinite number of series for $W(\ )$, since

$$= v_n e^{v_n} e^{p_n} \tag{79}$$

as can easily be established by induction. This gives us a family of bi-infinite sequences of series

$$W_\ell(\ ) = \sum_{k\geq 0} \frac{q_k(v_n + 2\pi i \mathcal{U}_\ell(v_n))}{(1 + v_n + 2\pi i \mathcal{U}_\ell(v_n))^{2k-1}} \frac{p_n^k}{k!} \tag{80}$$

for $W(\ )$. We may add a further infinity of series by using different branches for ln, by putting $v_0^{(m)} = \ln_m(\ )$ and $p_0^{(m)} = -\ln \ln_m$

More generally, for any b and c and any power series

$$G(x) = \sum_{i\geq 0} a_i x^i \tag{90}$$

with $a_0 \neq 0$, the following expansion holds:

$$e^{bW(z)}W(\ )^c G\left(\frac{1}{W(\ )}\right) = \ ^b \ln^{c-b} \ \sum_{n\geq 0} \frac{A_n(\ln\ln\ \ )}{\ln^n} \tag{91}$$

where the polynomials A_n are computable by $A_0 = a_0$, $A'_{n+1}(x) = (b-c+n)A_n(x) - A'_n(x)$ and $A_n(0) = a_n$.

4.11 Series for $W(\rho\exp(it))$

If $y = W(\rho\exp(it))$ for $\rho \geq 0$ and $-\pi < t \leq \pi$, then define v by

$$W(\rho e^{it}) = W(\rho) + it + v\,. \tag{92}$$

Then $W\exp W = \rho\exp(it)$ implies

$$(W(\rho) + it + v)e^{W(\rho)+it+v} = \rho e^{it} \tag{93}$$

or, using $W(\rho)\exp(W(\rho)) = \rho$,

$$1 + \frac{it}{W(\rho)} + \frac{1}{W(\rho)}v = e^{-v}\,. \tag{94}$$

But this is just (70) with $\sigma = 1/W(\rho)$ and $\tau = -it/W(\rho)$, and thus all of our fundamental series solutions to (70) apply! The nicest one is (75), which splits into separate series for the real and imaginary parts of $W(\rho\exp(it))$. We have

$$W(\rho e^{it}) = \sum_{n\geq 0} \frac{q_n(W(\rho))}{(W(\rho)+1)^{2n-1}}\frac{(it)^n}{n!} \tag{95}$$

and clearly all the odd terms are purely imaginary and the even terms are real.

5 Infinite Products

From the relation $W(\ ) = \ \exp(-W(\ ))$ it is easy to see that any series for $W(\ )$ may be trivially transformed into an infinite product. For example, from the series (6) we have

$$W(\ ) = \ \prod_{n\geq 1}\exp(\frac{(-n)^{n-1}}{n!}\ {}^n) \tag{96}$$

but this of course gives us no essentially new information. However, the series (83) gives us

$$W(\ ) = \ln(\ )\prod_{n=0}^{\infty}(1 + p_n/v_n) \tag{97}$$

in terms of the iterates of (76), and this is a simpler and more natural representation if nothing else.

6 A Final Pair of Expansions

The iterations (76–77) may be used to show that $W(\ )$ can be written as

$$W(\ ) = \frac{\qquad}{\exp\frac{z}{\exp\ \ }}$$

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