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RESEARCH ARTICLE

Bernstein, Pick, Poisson and related integral expressions for Lambert W

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The Lambert W function has a number of integral expressions, including integrals of Bernstein, Thorin, Poisson, Stieltjes, Pick and Burniston-Siewert types. We give explicit integral expressions for W for each of these types. We also give integrals for a number of functions containing W.

1. Introduction

The Lambert W function is a multivalued inverse of the mapping $W \mapsto We^W$. Its branches, denoted by W_k ($k \in \mathbb{Z}$), are defined through the equations [10]

$$\forall z \in \mathbb{C}; \qquad W_k(z)\exp(W_k(z)) = z\,; \tag{1}$$

$$W_k(z) \sim \ln_k z \text{ as } \Re z \to \infty\,; \tag{2}$$

Definition 2.1 *A function* $f : (0;\infty) \to \mathbb{R}$ *is called a* Stieltjes function *if it admits a representation*

$$f(x) = a + \int_0^\infty \frac{d\sigma(t)}{x+t} \quad (x > 0)\;, \tag{3}$$

where a is a non-negative constant and σ *is a positive measure on* $[0;\infty)$ *such that* $\int_0^\infty$

where the unknown function $\sigma(t)$ can be determined using the Stieltjes-Perron inversion formula [12, p. 591]

$$\sigma(t) = \frac{1}{\pi} \lim_{s \to 0+} \Im \int_{-\infty}^{t} W'(\tau + is)\, d\tau$$

for all continuity points on the t-axis. Since $\sigma(t)$ is defined up to an arbitrary constant, one can set, after integrating,

$$\sigma(t) = \frac{1}{\pi} \lim_{s \to 0+} \Im W(-t + is) = \frac{1}{\pi} \Im W_0(-t)\,, \tag{10}$$

where the limit uses the continuity from above of W on its branch cut. The same result can be obtained using one of Sokhotskyi's formulas [13, p. 138].

To verify that $\sigma(t)$ satisfies the conditions in Definition 2.1, we use [15, lemma 1.1] to trim the domain of integration in (9) to $1/e < t < \infty$. In addition, $\sigma(t)$ can be regarded as a positive measure such that $d\sigma(t)/dt = o(1/t)$ at large t. Therefore $\int_{1/e}^{\infty} (1+t)^{-1} d\sigma(t) < \infty$ as required. Thus (9) takes the form

$$W'(z) = \frac{1}{\pi} \int_{1/e}^{\infty} \frac{1}{z+t} \frac{d \Im W_0(-t)}{dt} dt\,. \tag{11}$$

Changing to the variable $v = \Im W_0(-t)$ as before, we obtain (8).

Remark 2 Formula (11) can also be found by considerations similar to those used in [15] to prove (5). Moreover, (11) is a result of differentiating (5) with subsequent integration by parts. The representation (11) is also found in [21].

Remark 3 Comparing formulae (5) and (11) shows that the latter can be formally obtained from the former by replacing the ratios $W(z)/z$ and $\sigma(t)/t$ respectively with the derivatives $dW(z)/dz$ and $d\sigma(t)/dt$, where $\sigma(t)$ is defined by (10).

Corollary 2.4

$$\int_0^{\pi} \left(\frac{\sin v}{v} e^{v \cot v} \right)^p dv = \frac{\pi p^p}{p!}\,, \quad p \in \mathbb{N}\,. \tag{12}$$

Proof The integral (9) can be written as

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^n}{n!} z^{n-1} = \frac{1}{\pi} \int_0^{\pi} \frac{dv}{z - t}\,, \tag{13}$$

where t is the same as in (6) and the left side is the Taylor series for W' and is convergent for $|z| < 1/e$. Since $|t| > 1/e$ and therefore $|z| < |t|$, we can expand $(z - t)^{-1}$ in non-negative powers of z. Equating the coefficients of like powers of z we obtain the equality

$$(-1)^{n-1} \frac{n^n}{n!} = -\frac{1}{\pi} \int_0^{\pi} \frac{dv}{t^n}\,,$$

which after substituting for t results in (12).

It is obvious that if the integral (12) is known, then going back from it to (13) we find (8). The integral (12) was conjectured by Nuttall for real $p \geq 0$ [20];

By [15, Lemma 1.2], the measure $=W(\ t)$ satisfies the requirements needed for [23, Theorem 8.2]. Changing to the variable $v = \ =W(\ t)$ as before and taking a holomorphic extension of the result to the cut z-plane $\mathbb{C}n(\ 1; 0]$ satisfying near conjugate symmetry, we obtain (22).

Remark 1 In the terminology of [23, p. 75], the integral form (23) is the Thorin representation of W function and $\ (t) = \ =W(\ t)=\ $ is the Thorin measure of W.

Remark 2 Differentiating the representation (22) for $W(z)$ gives formula (8) for $W^0(z)$.

Remark 3 The representation (23) (up to changing t to $\ t$) was obtained in [7] as a dispersion relation for the principal branch of W function. The representations (18)-(19) and (23) are also found in [21].

Note that by (19) function $'(\ )$ is completely monotonic, as it should be by Definition 3.2.

4. Pick representations

Definition 4.1 *[4, Definition 4.1] A function $f(z)$ is called a Pick function (or Nevanlinna function) if it is holomorphic in the upper half-plane $=z > 0$ and $=f \quad 0$ there.*

A Pick function $f(z)$ admits an integral representation [4, Theorem 4.4]

$$f(z) = \ _0 + b_0 z + \int_{\ 1}^{1} \frac{1 + tz}{(t \quad z)(1 + t^2)} d\ (t) \quad (=z > 0); \tag{24}$$

where

$$_0 = <f(i); \quad b_0 = \lim_{y\,!\,1} \frac{f(iy)}{iy}; \tag{25}$$

and $\ $ is a positive measure which satisfies

$$\lim_{s\,!\,0+} \frac{1}{\ } \int_{\mathbb{R}} =f(t + is)\,'(t)\,dt = \int_{\mathbb{R}} '(t)\,d\ (t) \tag{26}$$

for all continuous functions $' : \mathbb{R}\ !\ \mathbb{R}$ with compact support. The formula (24) with the integral written in terms of a measure $d\tilde{\ }(t) = \ (1 + t^2)^{\ 1} d\ (t)$ is called a *Nevanlinna formula* [17, p. 100].

Since $W(z)$ is a holomorphic function in the upper half-plane $=z > 0$, where $=W(z)W(zW$

where $\alpha_0 = \Re W(i) = 0.3746990\ldots$,

$$K(z;v) = \frac{(1 + zt(v))\left(v^2 + (1 - v\cot v)^2\right)}{(z - t(v))\,(1 + t^2(v))}\,; \tag{28}$$

and $t(v)$ *is defined by* (6).

Proof We apply formulae (25) and (26) to function $f(z) = W(z)$ to obtain

$$\alpha_0 = \Re W(i)\,; \quad b_0 = \lim_{y\to\infty} \frac{W(iy)}{iy}\,; \quad d\sigma(t) = \frac{1}{\pi}\Im W(t)\,dt\,.$$

Thus $b_0 = 0$, and since $\Im W(t) = 0$ for $t \geq -1/e$, we obtain

$$W(z) = \alpha_0 + \frac{1}{\pi}\int_{-\infty}^{-1/e} \frac{1 + tz}{(t - z)(1 + t^2)}\Im W(t)\,dt \quad (\Im z > 0)\,. \tag{29}$$

By the change of variable $v = \Im W(t)$ in the integral (29) (see (6)) we obtain formula (27), which is also valid in the lower half-plane $\Im z < 0$ in accordance with the near-conjugate symmetry of W.

Corollary 4.3

$$\frac{W(z)}{z} = \beta_0 \exp\left(-\frac{1}{\pi}\int_0^\pi K(z;v)t(v)dv\right) \quad (|\arg z| < \pi)\,; \tag{30}$$

where $\beta_0 = \exp(-\Re W(i)) = 0.6874961\ldots$.

Proof It immediately follows from (27) owing to the identity $W(z)/z = e^{-W(z)}$.

Now we take advantage of the fact that if f is a Stieltjes function then $-f$ and $1/f$ are Pick functions [4]. Therefore, $-W(x)/x$ and $x/W(x)$ are Pick functions that admit a representation (24).

Theorem 4.4 *For the principal branch of the W function the following formulae, with $K(z;v)$ defined by* (28), *hold.*

$$\frac{W(z)}{z} = \gamma_0 + \frac{1}{\pi}\int_0^\pi K(z;v)\,dv \quad (|\arg z| < \pi)\,; \tag{31}$$

$$\frac{z}{W(z)} = \delta_0 - \frac{1}{\pi}\int_0^\pi K(z;v)e^{-2v\cot v}\,dv \quad (|\arg z| < \pi)\,; \tag{32}$$

where $\gamma_0 = \Re\,[W(i)/i] = \Im W(i) = 0.5764127\ldots$, $\delta_0 = \Re[i/W(i)] = 1.2195314\ldots$.

The constants in (27) — (32) obey $\alpha_0 + i\gamma_0 = W(i)$, $\beta_0 = e^{-\alpha_0} = \gamma_0/\cos\gamma_0$, and $\delta_0 = \gamma_0/(\alpha_0^2 + \gamma_0^2)$.

We add in one more integral representation associated with the Nevanlinna for-represented as

$$f(z) = z \quad b + \int_0^{\infty} \frac{dh(r)}{z^2 + r} \quad ; \tag{33}$$

where constant $b \geq 0$ and

$$h(r) = \frac{2}{\pi} \lim_{x \to 0} \Re \int_0^{\sqrt{r}} f(x + iy)dy \, . \tag{34}$$

In fact, the formula (33) follows from the Nevanlinna formula (or (24)) after changing the variable $z \to iz$

Proof We consider the de ning equation (1) as an equation $F(W) = 0$ with respect to W, where

$$F($$

have cos ′ = 1 x=e and sin ′ = 0. Now ′

$$W(x) = \frac{1}{2\ i} \int \ln \frac{F(\ )}{} d\ ; \qquad (45)$$

where the integration contour is the unit circle $j\ j = 1$ and $x\ 2\ (\ 1{=}e; e)$. Since $F(0) = \ x$ and $W(x){=}x = e^{\ W(x)}$, formula (44) is simpli ed

$$W(x) = \frac{1}{2\ i} \int \frac{\ln(F(\ ){=}\ )}{} d\ : \qquad (46)$$

We set $= e^{i}\ ;$. Then, as $F(\ ){=}\ = F(e^{i}\ )e^{\ i} = R(\ ) + iI(\ )$, where

$$R(\ ) = 1 \quad xe^{\ \cos}\ \cos(\ + \sin\ );$$
$$I(\ ) = \quad xe^{\ \cos}\ \sin(\ + \sin\ );$$

and $d\ = \ = id$, the integral (46) is reduced to

$$W(x) = \frac{1}{2}\int_0 \ln\ R^2(\ ) + I^2(\ )\ d\ : \qquad (47)$$

Similarly, the integral (45) can be represented in the form

$$W(x) = \frac{1}{2}\int_0 \quad 2\arctan(I(\ ){=}R(\ ))\sin \quad \ln\ R^2(\ ) + I^2(\ )\ \cos\ \ d\ ; \qquad (48)$$

where we have taken into account that $\arg(R(\ ) + iI(\ )) = \arctan(I(\ ){=}R(\ ))$ as $R(\ ) > 0$ for $0 < \ <$ and $1{=}e < x < e$. We note that the integral (47) has a simpler form than (48). Integrals similar to the above with using a function $\bar{F}(\ ) = \ e \quad x$ in our notations instead of (40) in formulas (44) and (45) (without simpli cation (46)) are given in [1].

Thus the integrals (47) and (48) representing the principal branch of the Lambert W function are valid in the domain that contains interval $(\ 1{=}e; 0)$.

However, there is one more branch that is also a real-valued function on this interval, this is the branch 1 with the range $(\ 1;\ \ 1)$ (recall $W_0 > \ 1$ and $W_0(\ 1{=}e) = W_{\ 1}(\ 1{=}e) = \ 1$) [10]. To obtain a representation of this branch we nd a zero of function $(\ ) = 1 \quad xe \quad =$ in $(\ 1;\ \ 1)$ for xed x using an approach [2] (cf. formulae (12) and (8) therein)

$$W_{\ 1}(x) = \ c \quad \frac{1}{2\ i}\int_C \ln \frac{(\ )}{\ + c} d\ ;$$

where the circle C is de ned by equation $j\ + cj = c \quad 1$ with arbitrary constant $c > 1$ and $1{=}e < x < \ (2c \quad 1)e \quad 1$ with arbitrary constant where $\;=\;(\;)=1+2(c\quad 1)\sin^2(\;=2)$ and $\;=\;(\;)=(c\quad 1)\sin\;$. As a result, some applications. One of them has been mentioned in connection with finding Nuttall-Bouwkamp integral (12). Other definite integrals appear when taking particular values of z. For example, integrals (4), (14), (15), (35) taken at $z = e$ yield new integrals for .

$$= \int_0 \frac{v^2 + (1 \quad v\cot v)^2}{1 + v\csc v e^{(1+v\cot v)}} dv\,;$$

$$= \int \frac{v\,dv}{v + e^{1+v\cot v}\sin v}\,;$$

$$= \frac{e}{e \quad 1} \int_0 \frac{v^2 + (1 \quad v\cot v)^2}{v\csc(v)\,(v\csc(v) + e^{1+v\cot v})} dv\,;$$

$$= \int_{=2}^{=2} v^2 + (1 + v\tan \mathrm{v})\,($$

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