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Chapter 4

Some basic principles in interaction calc[redacted]lations

D.J. Jeffrey

Department of Applied Mathematics, The Universit of Western Ontario, London, Ontario, Canada N6A 5B7

Abstract

Some general aspects of the interactions taking place within a suspension can be understood by exploiting properties of the Stokes equations; these properties and their applications are described. Specifically, the way in which reversibility can be used to predict overall properties of a flow is explained by analysing several applications both informally and formally; Faxén's laws for the response of a particle to an ambient flow are examined to clarify common conceptual difficulties; and the use of lubrication theory to approximate interactions between close particles is developed carefully. In addition, the ways in which tensors can be used to summarize interaction results are covered, and the principles are illustrated whereby tensor relations can be simplified by appealing to geometrical symmetries and the character of tensor transformations.

Nomenclature

a	Radius of spherical or circular particle
A, B	Resistance tensors
E	Rate-of-strain tensor
$\boldsymbol{f}$	Equivalent surface forces
$\boldsymbol{F}, F_i$	Force on particle
$\boldsymbol{g}$	Acceleration due to gravity
G	Green's function for Stokes flow
G_{ijk}	Resistance tensor

h

about the interactions bet een suspended particles at lo Reynolds numbers than it is at higher Reynolds numbers obtained its position and velocity; the same flo ill be found hether the sphere is moving at constant velocity $\boldsymbol{V}$ or hether it is undergoing an oscillatory motion and has the velocity $\boldsymbol{V}$ only at that instant. Consequently, if time is reversed, there is no change in the governing equations, and the system retraces its steps, moving in a valid Stokes flo .

Some visually compelling demonstrations of reversibility can be seen in the film *Low-Reynolds-number flows* that as made by G.I. Taylor in 1967*. One demonstration, for instance, uses very viscous oil in the annulus bet een t o concentric vertical cylinders, the inner cylinder being able to rotate about its vertical axis. Some dye is injected into the oil and then the cylinder is rotated, causing the dye to smear out, or so it seems. When the motion is reversed, ho ever, all the dye returns to its starting position (except for a little blurring due to molecular diffusion). A similar demonstration, using the same apparatus, places a rigid body in the oil, and again the inner cylinder is rotated. The body translates and rotates a ay from its initial configuration, but hen the all motion is reversed, the body returns to its starting point. Another demonstration sho s a small mechanical fish vainly trying to s im by flapping a tail. At high Reynolds number, it moves for ard, but at lo Reynolds number, each time the tail reverses its motion, so does the fish.

The above demonstrations require reversibility alone; the next one combines reversibility ith another symmetry possessed by the flo . Consider a sphere falling parallel to a vertical plane all. The fact that the sphere falls at a constant distance from the all is demonstrated in the follo ing ay. First one imagines that the sphere falls for a short time do n ards. For the sake of setting up a contradiction, it is necessary to conjecture that it is *not* the case that the sphere stays at a constant distance from the all, and that instead it moves a ay from the all. No consider hat ould happen if time ere to run back ards. Clearly the sphere ould retrace its path and move closer to the all as it rose. No observe that a second ay to get the sphere to move up ards is to reverse gravity hile leaving time running for ard. Because the all is a plane, the flo situation after gravity has been reversed is identical to the situation before (notice that this geometrical symmetry is independent of the reversibility just described). Accordingly, if the sphere is subjected to an up ard force, then, by the starting conjecture, it ill move a ay

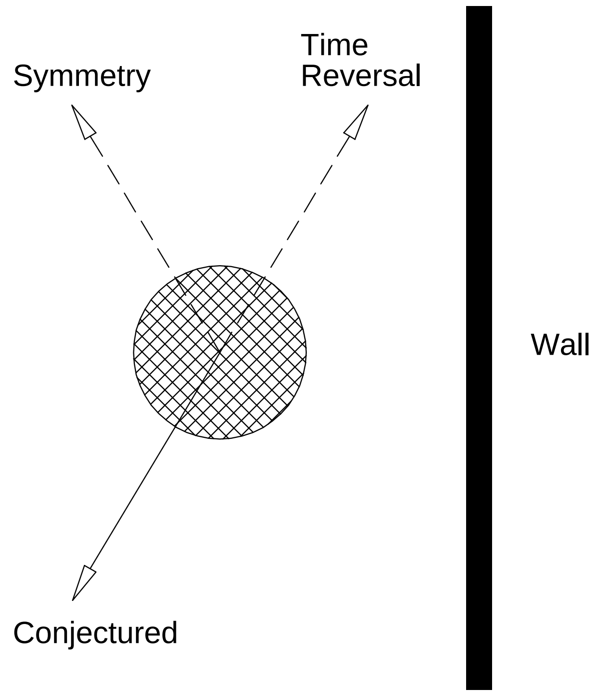

Figure 1. A sphere falling, or rising, next to a plane all.

from the all as it rises, as sho n in figure 1.

This is a contradiction: on the one hand, geometrical symmetry says that an up ard-moving sphere ill move a ay from the all, hile reversibility says that the sphere ill move to ards the all. The conclusion is that it ill neither move closer nor a ay, but stay at the same distance.

Reversibility is also important for periodic motion. Numerous periodic motions have been found in systems of particles moving in Stokes flo , and their existence can have a strong influence on calculations. If the system is scleronomic, meaning the external forces on the flo do not change ith time, then a moving system of particles that passes through a configuration t ice must be executing a periodic motion, and reversibility combined ith another symmetry can prove this, as the tumbling of an ellipsoidal particle in a shear flo illustrates. The ellipsoid executes a closed periodic orbit, and this orbit is described by an orbital constant, hose value depends upon ho the particle starts its motion. One might suppose that the ellipsoid ould follo an orbit that is not exactly peri-

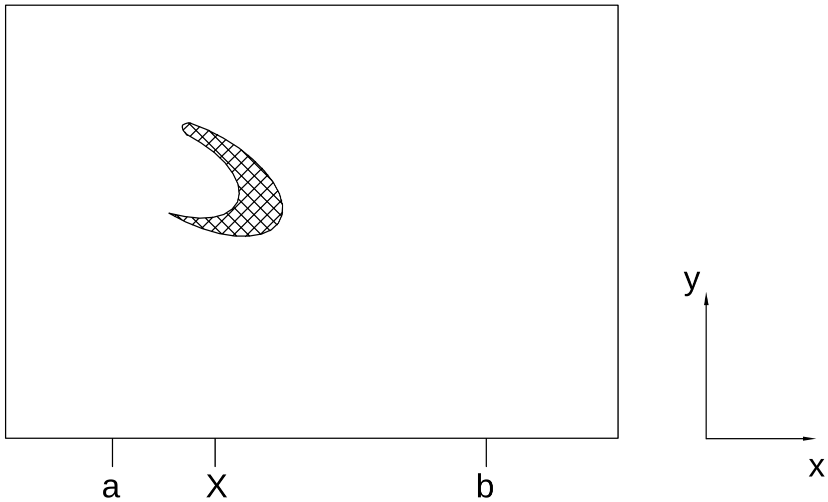

Figure 2. A simplified version of Taylor's demonstration.

ith respect to its centroid at $\boldsymbol{x}_c$. The boundary condition on its surface is then

$$\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{\Omega} \times (\boldsymbol{x} - \boldsymbol{x}_c) \ , \tag{2}$$

and in addition $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ are chosen so that the total force $\int \boldsymbol{\sigma} \cdot \boldsymbol{n} \ \mathrm{d}S$ on the particle is zero, as is the total moment $\int \boldsymbol{\sigma} \cdot \boldsymbol{n} \times (\boldsymbol{x} - \boldsymbol{x}_c) \ \mathrm{d}S$. Here $\boldsymbol{\sigma}$ is the stress tensor. It ill no be proved that, for the purpose of calculating the displacements of fluid particles, it does not matter ho the point X gets from $x = a$ to $x = b$; in other ords, for all possible functions $X(t)$, the fluid elements and the suspended particle [illegible] and the su V Ti [illegible] condition is also satisfied, because, using subscript notation,

$$\begin{aligned}\sigma_{ij} &= -p\delta_{ij} + \mu\left[\frac{\partial u_i}{x_j} + \frac{\partial u_j}{x_i}\right] \\ &= V(t)\left(-p^*\delta_{ij} + \mu\left[\frac{\partial u_i^*}{x_j} + \frac{\partial u_j^*}{x_i}\right]\right) = V(t)\sigma_{ij}^* \ .\end{aligned}$$

Since σ_{ij}^* integrates to zero force, so does σ_{ij}. The other boundary conditions are satisfied in a similar manner.

The displacement of a fluid element is calculated by considering hat happens during a time δt. If a fluid element is at $\boldsymbol{x}_L(t)$ (the subscript is a reminder that $\boldsymbol{x}_L$ is a Lagrangian quantity), then its motion is given by

$$\frac{\mathrm{d}\boldsymbol{x}_L}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}_L(t), t) \ , \tag{3}$$

here $\boldsymbol{u}$ is the Eulerian velocity that obeys the Stokes equations. Therefore, in time δt, the displacement of the fluid element is given approximately by $\delta t\, \boldsymbol{u}(\boldsymbol{x}_L, t) = \delta t\, V(t)\boldsymbol{u}_L^*$, using the results established above. No $\delta t\, V(t) = \delta x$, the displacement of the boundary. Thus all fluid displacements during the motion are proportional to the corresponding displacement of the boundary, implying that the total displacement of any particular element is proportional to $\int_0^T V(t)dt = X(T) - X(0) = b - a$, hatever the function $X(t)$. This can be seen in the demonstrat e part of the solution), together ith $\boldsymbol{t} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} = 0$ here $\boldsymbol{t}$ is a unit tangent vector to the surface. The equations and boundary conditions are again linear and scale ith the boundary velocity.

The strong form of reversibility applies to systems in hich the Stokes equations contain a single time scale, imposed by the boundaries through the boundary conditions. Such cases ill agile time sc n T ound Ti c r ersi Ti n b T r Ti containing very small particles ill be subject to physical processes that modify the governing equations and destroy reversibility, such as electric charge effects, Bro nian motion, and so on.

3 Faxén laws

Faxén la s are exact mathematical statements about the response of a particle to an ambient flo . They are used, ho ever, mostly in the approximate calculation of interactions bet een particles, and this ambivalence can be a cause of confusion. Suppose that a region of fluid contains an ambient flo field $\boldsymbol{u}^{\infty}(\boldsymbol{x})$, and suppose that a rigid spherical particle is introduced into this field at a point $\boldsymbol{x} = \boldsymbol{x}_0$. The force $\boldsymbol{F}$ and couple $\boldsymbol{L}$ experienced by the sphere, hich is stationary, are given exactly by

$$\boldsymbol{F} = 6\pi a\mu \left[\boldsymbol{u}^{\infty}(\boldsymbol{x}_0) + \tfrac{1}{6}a^2\nabla^2\boldsymbol{u}^{\infty}(\boldsymbol{x}_0)\right] \ , \tag{5}$$

and

$$\boldsymbol{L} = 8\pi a^2\mu\nabla\times\boldsymbol{u}^{\infty}(\boldsymbol{x}_0) \ . \tag{6}$$

La s for other quantities are also kno n. It is certainly striking, on first acquaintance, to see that only the local velocity and its first and second derivatives are important, no matter ho complicated the ambient flo may be. As Kim & Karrila (1991) point out, the interpretation can be simplified further by noting that $\nabla^2\boldsymbol{u}^{\infty}$ is proportional to the ambient pressure gradient. It is natural to contrast this simplicity ith the lengthy calculations in the literature of the motion of a particle near a all or a second particle, and to onder ho it can be exact. Students sometimes dra a diagram of a sphere in a maelstrom of streamlines, and ask hether Faxén la s applies to this situation.

An understanding of hether Faxén la s are exact or approximate can be obtained by revie ing a proof of them, here the proof that as given in the appendix to Batchelor (1972). Any sol[illegible]ion of the ‘ B[illegible]

Then $\mathsf{G}(\boldsymbol{x}-\boldsymbol{\xi})\cdot(\boldsymbol{n}\cdot\boldsymbol{\sigma}\ \mathrm{d}S)$ is the velocity field produced at $\boldsymbol{x}$ by that force. The second term $\boldsymbol{n}\cdot$ [illegible] $\cdot\,\boldsymbol{u}\ \mathrm{d}S$ corresponds to a 'double layer', and has a similar interpretation. The solution (7) can be interpreted by saying that the flo outside a rigid particle cannot tell the difference bet een a physical boundary and an appropriate distribution of point forces and double layers. Faxén's theorem is derived by adding such a distribution of forces to a region of fluid so

Some of the usual questions about Faxén la s can no be ans ered. The first question asks 'Is it really exact?'. The ans er is 'Yes', provided the limit $R \to \infty$ is accepted. This limit is often taken in fluid mechanics, and although it can cause difficulties, for example uniform flo around a cylinder in t o dimensions, it is usually accepted if $\boldsymbol{u}^\infty$ is simple. The second question asks 'Does it apply to flo s ith large curvature?'. Large curvature means that the values of $\nabla \boldsymbol{u}^\infty$ and higher derivatives are large, in order for the streamlines of $\boldsymbol{u}^\infty$ to be highly curved. Here the ans er is 'Technically yes, but in practice no'. It is technically yes, because it is possible mathematically to imagine a flo ith large curvature produced by very distant boundaries. It is no in practice, because the person asking the question is almost certainly thinking of a situation in hich the curvature is caused by boundaries fairly close to the particle. For example, the particle is in a curving pipe or near other particles. Thus the flo is probably not caused by large forces far a ay, but by ordinary forces that are close. In this case the questioner is really refusing to ignore the reflections from the boundary, in other ords, refusing to accept the premise of the theorem.

4 Lubrication theory and close particles

The interactions bet een t o nearly touching particles can often be calculated using *lubrication theory*. This 'theory' is actually a set of approximations that allo the Navier-Stokes equations to be simplified to a form in hich they are more easily solved. In order for the approximations to be valid, the particles must be nearly touching, and in addition their relative motion must cause the fluid in the gap bet een them to be highly sheared. For example, if t o close particles are in the process of moving past each other, then lubrication theory can be applied, but if they are sedimenting side-by-side ith no relative motion bet een them, it cannot. Although the name lubrication theory is a reminder that the approximations ere first orked out in the analysis of flo s in lubricated bearings, the theory in the lo -Reynolds-number context is not identical to its engineering namesake; some of the ays in hich the approximations are developed in Stokes flo give the application a distinctive slant.

The aspects of the theory that ill be explored are the definition of the gap bet een the particles, the handling of the edge of the gap, and the role of the fluid far from the gap. It is important to discuss these points

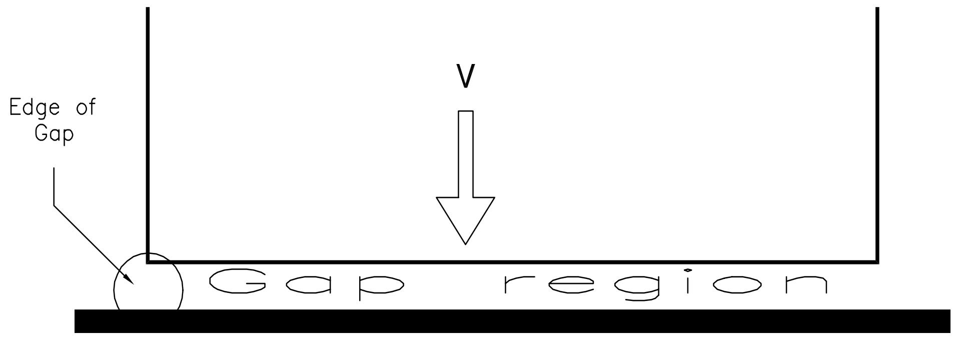

Figure 3. A plate approaching a all.

because the justification for the theory is usually informal; it is developed initially for a situation in hich the approximations have a strong intuitive appeal, or for a situation in hich they can be proved correct. Subsequently, the approximations are applied to situations in hich the justification is less straightfor ard, and there is a danger that future calculations ill continue to use them hen they have really ceased to applydas l n sit Vn Ti i P non-dimensionalized velocity components be given by $\boldsymbol{u} = V(u, v)$. If the pressure is given by $\mu V p$, the equations are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} , \tag{10}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial p}{\partial y} , \tag{11}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \tag{12}$$

The no-slip boundary conditions require that

$$u(x,0) = u(x,h) = 0 \quad \text{and} \quad v(x,0) = 0 \quad \text{and} \quad v(x,h) = -1 .$$

Since, from the boundary conditions, v goes from -1 to 0 in a distance h, the term $\partial v/\partial y$ ill be approximately $-1/h$, and then, from the continuity equation (12), this implies that $\partial u/\partial x \approx 1/h$. This in turn implies $u \approx x/h$, hich is consistent ith our global estimate above. Since u gro s ith x hile v does not, it must be that $u \gg v$ for $x \gg h$, i.e. a ay from the centre of the gap. So the terms in (11) are much less than those in (10), and (11) can be neglected. Further, $\partial p/\partial y$ can be set to zero in comparison ith $\partial p/\partial x$, meaning that p is approximately independent of y. The assumption that pressure is approximately constant across a thin layer of fluid is also used in boundary-layer theory. In equation (10), the derivative $\partial^2 u/\partial y^2$ must be of the order of Vx/h^3, since u changes from 0 at the alls to Vx/h in the flo . The estimate $u \approx Vx/h$ also implies $\partial^2 u/\partial x^2 \approx 0$, so clearly $\partial^2 u/\partial y^2 \gg \partial^2 u/\partial x^2$, and therefore (10) can be approximated by

$$\frac{\partial^2 u}{\partial y^2} = \frac{\mathrm{d}p}{\mathrm{d}x} , \tag{13}$$

the total derivative of p sho ing that it depends only on x. Inte

Therefore $p = -6x^2/h^3 + Ax + B$. The calculation is completed by applying boundary conditions on p. By the symmetry of the problem, $A = 0$, but hat about B, the pressure at the centre of the gap?

As the plate approaches the plane, the fluid from the gap region ill escape into the surrounding fluid, spreading out until it reaches the ambient pressure. Depending upon the Reynolds number of the flo outside the gap, this could even be a jet-like flo . The problem of ho the fluid slo s do n on leaving the gap is a difficult calculation and it is better avoided if possible. Consequently, it is assumed that as soon as the fluid reaches $x = \pm L$, the pressure equals the pressure outside the gap (hich is zero). This gives $p = 6(L^2 - x^2)/h^3$, and a non-dimensionalized force of $2\int_0^L p \; \mathrm{d}x = 8L^3/h^3$. The last assumption is in fact the specification of the edge of the gap (hich as left vague above) and the reader's acceptance of it depends to some extend on mathematical outlook. The clear-cut geometry suggests strongly that the pressure ill reach the ambient one ithin a distance $O(h)$ from the edge, that is, ithin the circle labelled 'edge of gap' on figure 3. This ill induce an error $O(h/L)$, hich is of the same order as the approximations made in obtaining (13). It is possible to contrive flo conditions outside the gap that ould invalidate the assumptions about the edge of the gap, but it is not a serious orry and the approximations above are ell established.

The above example is important in engineering lubrication theory, but for sedimentation studies and other particulate interaction problems, the geometry is only generally relevant. Keeping ith the simplification of t o-dimensions, consider a cylinder approaching a plane, again ith velocity V, as sho n in figure 4. From looking at the figure, one can be convinced that there is a gap region bet een the cylinder and plane, although it is no longer possible to point to the edge of the gap ith the confidence felt in the first example. One proceeds to analyse the flo in the gap, in the hope that everything ill turn out all right in the end.

Let the radius of the cylinder be a and let the gap be h at its minimum. The velocities and pressure are again $V(u, v)$ and $\mu V p$ and the approximations leading to (12) and (13) still apply. The boundary conditions can be simplified by expanding the expression for the cylinder surface for small values of x. Thus

$$y = a + h - \sqrt{a^2 - x^2} = h + \frac{x^2}{2a} + O(x^4) = H + O(x^4) \; , \qquad (15)$$

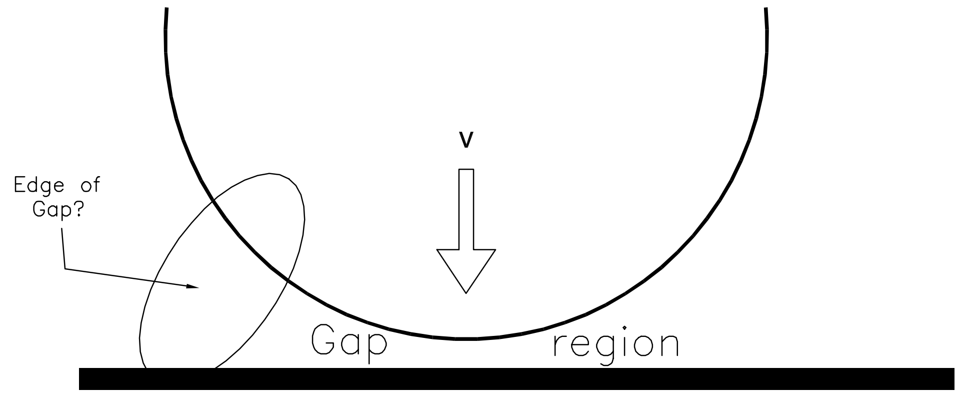

Figure 4. A

in some presentations, this limit is taken early in the treatment, so that the edge of the gap never appears.

The lubrication result (19) can be compared ith the exact result obtained by Jeffrey & Onishi (1981). When their solution is expanded for small h, it becomes

$$F = 3\sqrt{2}\pi \left(\frac{h}{a}\right)^{-\frac{3}{2}} + \frac{63\sqrt{2}}{20}\left(\frac{h}{a}\right)^{-\frac{1}{2}} + O\left(\left(\frac{h}{a}\right)^{\frac{1}{2}}\right) . \qquad (20)$$

Both (19) and (20) contain the same leading term, but (20) contains a second singular term. Numerically, both terms are important: if $h = 0.1$, then the force according to the exact solution is 148.31, hereas one term of (15) gives 134.2 and t o terms give 148.25. So the comparison provides reassurance on one point, but raises another. In particle geometries, the edge of the gap can be avoided to leading order, but lubrication theory must be extended to higher order to capture all of the important terms. Can the edge of the gap be ignored at higher order? This question has been investigated in three-dimensional flo s.

The change from t o dimensions to three dimensions 1 in nTi an Tn in the ree Ti i 7 Ts

The edge of the gap appears in the expression for the force in a ay that cannot be removed by taking the limit $R_G \to \infty$, and this is the feature that this exampl

On of the attractions of lubrication theory is the fact that it can yield important information about interactions from a relatively simple analysis of only one part of the flo . If the edge of the gap cannot be removed from the calculations, lubrication theory loses some of its appeal. Thus the importance of the O'Neill & Ste artson result lies in the fact that it provides a foundation for other lubrication calculations to dra on. Even if the edge of the gap remains in the result of a gap calculation, it can be removed by postulating that a full analysis ould find a cancelling term in the solution outside.

In a number of publications, lubrication theory has been pushed even further. Thus the approximations above have been cab found good agreement ith experiment. The motion of one sphere approaching a much larger one has been follo ed very accurately by Lecoq

The need for axial vectors is demonstrated by considering a rotating rigid body in a very simple case: a rigid body rotating around the z axis ith vector angular velocity $\boldsymbol{\omega} = \Omega\boldsymbol{k}$. Consider velocities in the XY plane. The velocity (v_x, v_y) at any point (x, y) is given by $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, meaning its components are given by

$$v_x = -\Omega y \ , \qquad v_y = \Omega x \ . \tag{31, 32}$$

First consider the transformation to a dashed coordinate system defined by $x' = y$, $y' = -x$ and $z' = z$, hich is a rotation about the z axis. The matrix $a_{ij}^{(1)}$ for this transformation is

$$a^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{33}$$

The matrix $a^{(1)}$ can be used to transform both $\boldsymbol{\omega}$, hich remains $\Omega\boldsymbol{k}$, and $\boldsymbol{v}$ hose components in the ne coordinates become $v'_x = v_y$ and $v'_y = -v_x$. No consider the transformation to another ne , doubly dashed, coordinate system defined by $x'' = x$, $y'' = y$ and $z'' = -z$. The reader ill notice that the ne system is left handed. The matrix $a_{ij}^{(2)}$ for this transformation is

$$a^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ . \tag{34}$$

Clearly, this transformation causes the z component of any vector to change sign, since anything pointing along the old positive z axis must no point along the negative z'' axis. Suppose that this applies to the quantity $\boldsymbol{\omega}$. The transformation $\omega' = -\omega = -\Omega$ ould lead to a difficulty, because the velocity components for $\boldsymbol{v}$ ould become

$$v'_x = -\omega' y' = \Omega y \quad \text{and} \quad v'_y = \omega' x' = -\Omega x \ .$$

Thus the x and y velocities are no the negative of hat they ere before, although those axes ere unchanged. The only ay to correct this is to introduce a negative sign some here in the chain of definitions, and the standard place is the transformation la for $\boldsymbol{\omega}$. Thus the equation $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$ is kept intact (as is the definition of cross product in terms of components) as is the fact that $\boldsymbol{v}$ and $\boldsymbol{r}$ transform according to the vector la . No , ho ever, $\boldsymbol{\omega}$ transforms according to a pseudovector la , namely $\omega_i' = a_{ij}^{(1)}\omega_j$, but $\omega_i'' = -a_{ij}^{(2)}\omega_j$. This is the difference bet een ordinary, or polar, vectors such as velocity, force and position, and axial vectors such as angular velocity, and angular momentum.

The difference bet een vector types

every point is doubled also. More can be seen, ho ever, if the equation is re ritten using the results of the previous section.

$$\boldsymbol{u} = \left[\mathbf{I} \left(\frac{3}{4r} + \frac{1}{4r^3} \right) + \frac{\boldsymbol{xx}}{r^2} \left(\frac{3}{4r} - \frac{3}{4r^3} \right) \right] \cdot \boldsymbol{U} \ . \tag{39}$$

The ne feature of this equation is the fact that the brackets do not contain $\boldsymbol{U}$; the effects of the velocity of the sphere and the geometry of the sphere have been separated in the presentation of the equation.

To put it another ay, in one dimension a scalar quantity α depends linearly on a scalar quantity β hen $\alpha = k\beta$, and moreover the coefficient of proportionality k is also scalar. In (39) there is a similar linear relationship $\boldsymbol{u} = \mathbf{K} \cdot \boldsymbol{U}$, only no the linking coefficient has become a tensor, because $\boldsymbol{u}$ and $\boldsymbol{U}$ are not in the same direction (as a scalar coefficient ould imply).

A second example is provided by the drag on a non-spherical particle. The force $\boldsymbol{F}$ is linearly related to the velocity of the particle $\boldsymbol{U}$, but not necessarily in the same direction, as is sho n by riting $\boldsymbol{F} = \mathbf{A} \cdot \boldsymbol{U}$. The tensor $\mathbf{A}$ is called a resistance tensor and it contains only geometrical factors and information about the particle shape, but nothing about the velocity. This is some help, but $\mathbf{A}$ still contains 6 scalar components (not 9, because the reciprocal theorem sho s it is symmetric), and could be a difficult object to ork ith. If, ho ever, the particle is axisymmetric, $\mathbf{A}$ can be simplified further. Suppose the axis of symmetry is $\boldsymbol{d}$ (a unit vector), then

$$\mathbf{A} = K_1 \boldsymbol{dd} + K_2 \mathbf{I} \ , \tag{40}$$

here K_1 and K_2 are scalars based on the shape. The proof first extracts the component of velocity parallel to $\boldsymbol{d}$. From above, this is $\boldsymbol{U} \cdot \boldsymbol{dd}$. By symmetry, the force is parallel to this component, so $\boldsymbol{F} =$

linear in these quantities (cf. equation (2) above), the force is a l most general form for G_{ijk} is then

$$G_{ijk} = K_4 d_i d_j d_k + K_5 d_i \delta_{jk} + K_6 d_j \delta_{ik} + K_7 d_k \delta_{ij} \ . \tag{43}$$

Since E_{jk} is traceless, the term $K_5 d_i \delta_{jk}$ cannot contribute to F_i and may therefore be dropped. Similarly, because E_{jk} is symmetric, the terms $K_6 d_j \delta_{ik}$ and $K_7 d_k \delta_{ij}$ ill al ays contribute to F_i in the form $K_6 + K_7$, so one term could be dropped. Ho ever, for reasons of asthetics, they are more commonly set equal. Thus G_{ijk} is reduced to depending on t o independent scalar functions. As in the case of **A**, there is some arbitrariness in assigning the t o functions, cf. equations (40) and (41), and different authors might vary in their choices.

References

Aris,

Keller, J.B. (1 63) Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or non-conducting cylinders, *J. Appl. Phys.* **34**, 1– 3.

Kim, S. & Karrila, S.J. (1 1) *Microhydrodynamics*, Butterworth.

Kocabiyik, S. & Jeffrey, D.J. (1 4) Asymptotic analysis of interactions between highly conducting cylinders, *Appl. Math. Lett.* **7**, 5 –63.

Leal, L.G. (1 80) Particle motions in a viscous fluid, *Ann. Rev. Fluid Mech.* **12**, 435–476.

Lecoq, N., Feuillebois, F., Anthore, R., Petipas, C., & Bostel, F. (1 5) *J. Phys. II France,* **5**, 323–334.

Lorentz, H.A. (18 6) A general theorem concerning the motion of a viscous fluid and a few consequences derived from it, In: *Collected Papers, Volume IV,* pp7–14. Martinus Nijhoff, 1 37.

Morse, P.M. & Feshbach, H. (1 53) *Methods of Mathematical Physics*, McGraw-Hill.

O'Neill, M.E. & Stewartson, K. (1 67) On the slow motion of a sphere parallel to a nearby plane wall, *J. Fluid Mech.* **27**, 705.

Taylor, G.I. (1 67) *Low-Reynolds-Number Flows*, 16mm colour sound film, produced by Educational Services Inc.

Trahan, J.F. & Hussey, R.G. (1 85) The Stokes drag on a horizontal cylinder falling toward a horizontal plane, *Phys. Fluids,* **28**, 2 61–2 67.

Tory, E.M. & Kamel, M.T. (1 2) Note on the periodic motion of four spheres, *Powder Technol.* **73**, 5 – 6.

Tory, E.M., Kamel, M.T. & Tory, C.B. (1 1) Sedimentation of clusters of identical spheres. III. Periodic motion of four spheres, *Powder Technol.* **67**, 71.